Linear Algebra of Magic Squares

Michael Lee
Central Michigan University
Mt. Pleasant, MI 48859
email: lee4m@cmich.edu

Elizabeth Love
Howard University
Washington, DC 20001
email: lle_17@hotmail.com

Elizabeth Wascher
Central Michigan University
Mount Pleasant, MI 48859
email: wasch1ea@cmich.edu

Advisor: Dr. Sivaram K. Narayan
Department of Mathematics
Central Michigan University
Mount Pleasant, MI 48859
email: sivaram.narayan@cmich.edu

Abstract

A *magic square* M is an n-by-n array of numbers whose rows, columns, and the two diagonals sum to μ called the magic sum. If all the diagonals including broken diagonals sum to μ then the magic square is said to be *pandiagonal.* A *regular* magic square satisfies the condition that the entries symmetrically placed with respect to the center sum to $\frac{2\mu}{n}$. If the entries of M are integers 1 through n^2 the magic sum μ is $\frac{n(n^2+1)}{2}$ and M is said to be a classical magic square.

In this paper, we find vector space dimension of regular and pandiagonal magic squares. We give a simpler proof of the known result that even order regular magic squares are singular. We present matrix theoretic constructions that produce odd order regular magic squares that are singular and nonsingular.

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1 Introduction

An n-by-n *semi-magic square* is an n-by-n matrix of complex numbers in which the sum of entries along each row and each column is a constant. If in addition the sum of the entries along the main diagonal and the cross diagonal is the same constant then the n-by-n matrix is said to be a *magic square* with *magic sum* μ. If the elements of the magic square are consecutive integers from 1 through n^2 then it is called a *classical* magic square with magic sum $\mu = \frac{n(n^2+1)}{2}$. In this paper we are concerned with two special classes of magic squares.

Definition A magic square is said to be *pandiagonal* if the entries along all diagonals including broken diagonals sum to the constant μ.

Definition A magic square $M = [m_{ij}]$ is said to be *regular* if $m_{ij} + m_{n+1-i,n+1-j} = \frac{2\mu}{n}$ where μ is the magic sum. In other words, the sum of any two entries of M that are symmetrically placed across the center of the square is equal to $\frac{2\mu}{n}$.

Magic squares have fascinated people since 2200 B.C. See [2] to read about the history of magic squares. In this work, we focus on the linear algebraic properties of pandiagonal and regular magic squares. We present results on vector space dimension of regular and pandiagonal magic squares in section 2. In section 3 we present some known results on eigenvalues of magic squares (some with different proofs) to make this paper as self contained as possible. New results on regular magic squares can be found in sections 4-6. A simpler proof of the known result that even order regular magic squares are singular in given in section 4. A necessary and sufficient condition for nonsingularity of odd order regular (section 4), a construction of singular magic squares (section 5) and nonsingular regular magic squares (section 6) are given.

2 Vector Spaces of magic squares

Since the sum of two magic squares is a magic square and a scalar multiple of a magic square is a magic square, we see that the set of magic squares with complex entries is a subspace of the vector space of n-by-n complex matrices. If the magic sum of M is zero, M is said to be a zero magic square. Moreover, corresponding to each magic square A with magic sum μ there is an *associated zero magic square* M given by $M = A - \frac{\mu}{n}E$, where E is the n-by-n matrix with all entries equal to 1. It is easily observed that the set of n-by-n zero magic squares forms a subspace. The following theorem can be found in [12]. A similar result for entries from an arbitrary field can be found in [11].

Theorem 2.1. *[12] The dimension of the vector space of n-by-n zero magic squares is $n^2 - 2n - 1$.*

In this section we prove similar results about the dimension of n-by-n regular magic squares and n-by-n pandiagonal magic squares.

Lemma 2.2. *The set of n-by-n regular magic squares and the set of n-by-n pandiagonal magic squares form a vector space.*

Proof. Since n-by-n magic squares form a subspace of the vector space of n-by-n complex matrices, it easily follows from definition that addition and scalar multiplication preserve the property of being a pandiagonal or a regular magic square. ☐

We denote the n-by-n zero regular magic squares as $\emptyset RM(n)$ and the n-by-n zero pandiagonal magic squares as $\emptyset PM(n)$.

Theorem 2.3. *The dimension of $\emptyset RM(n)$ is $\frac{(n-1)^2}{2}$ when n is odd and $\frac{n(n-2)}{2}$ when n is even.*

Proof. Let $A = [a_{ij}]$ be a zero regular magic square. Suppose n is odd. Then one of the n^2 entries, namely $a_{\frac{n-1}{2},\frac{n-1}{2}} = 0$. Hence we consider a system of homogeneous linear equations in $n^2 - 1$ variables $(a_{11}, a_{12}, ..., a_{1n}, a_{21}, ..., a_{2n}, ..., a_{n1}, a_{n2}, ...a_{nn})$. There are $\frac{n^2-1}{2}$ equations giving the regularity conditions (starting with a_{11}), the n row sum conditions (starting with the first row), the n column sum conditions (starting with the first column), and two diagonal conditions. The system has $\frac{n^2-1}{2} + 2n + 2$ equations written in the order mentioned above. When $n = 3$, the 12-by-9 coefficient matrix appears as follows where the rows of the coefficient matrix are the conditions of the regular magic square and the j^{th} column of the coefficient matrix gives the coefficients of the j^{th} variable in the list a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}. Note that $a_{22} = 0$ and hence its coefficients in the fifth column are all zeros.

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}$$

In the coefficient matrix determined by the magic square A, the $\frac{n^2-1}{2}$ regularity conditions are clearly linearly independent. The sum of the first n regularity conditions equals the sum of the first row condition and the last row condition, namely rows $\frac{n^2+1}{2}$ and $\frac{n^2+2n-1}{2}$ of the coefficient matrix. Thus row $\frac{n^2+2n-1}{2}$ is a linear combination of the first n rows and row $\frac{n^2+1}{2}$ of the coefficient matrix. Similarly row $\frac{n^2+2n-3}{2}$ is a linear combination of rows $n+1$ to $2n$ and row $\frac{n^2+3}{2}$ of the coefficient matrix. Continuing this process, we show that each of the rows i such that $\frac{n^2+n+2}{2} \leq i \leq \frac{n^2+2n-1}{2}$ is a linear combination of n rows of regularity conditions and row $n^2 + n - i$. Moreoever, row $\frac{n^2+n}{2}$ of the coefficient matrix is the sum of the last $\frac{n-1}{2}$ of regularity conditions. Therefore $\frac{n^2-1}{2}$ regularity conditions and the first $\frac{n-1}{2}$ row conditions are easily seen to be linearly independent. A similar argument shows that each of the last $\frac{n+1}{2}$ column conditions are linearly dependent on n regularity conditions and one of the column conditions from the remaining $\frac{n-1}{2}$ column conditions. Thus the rank of the coefficient matrix is $\frac{n^2-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} = \frac{n^2+2n-3}{2}$. From the rank and nullity theorem we get the dimension of solution space to be equal to $n^2 - 1 - \frac{n^2+2n-3}{2} = \frac{n^2-2n+1}{2}$. Hence the dimension of $\emptyset RM(n)$ is $\frac{(n-1)^2}{2}$ when n is odd.

The case where n is even is similarly proved. Let $A = [a_{ij}]$ be a zero regular magic square. Suppose n is even. We consider a system of homogeneous linear equations in n^2 variables as above. There are $\frac{n^2}{2}$ equations giving the regularity conditions (starting with a_{11}), the n row sum conditions (starting with the first row), the n column sum conditions (starting with the first column), and the two diagonal conditions. The system has $\frac{n^2}{2} + 2n + 2$ equations written in the order mentioned above. When $n = 4$, the 18-by-16 coefficient matrix appears as follows:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}.$$

In the coefficient matrix determined by the magic square A, clearly the first $\frac{n^2}{2}$ regularity conditions are linearly independent. The sum of the first n regularity conditions equals the sum of the first row condition and the last row condition, namely rows $\frac{n^2+2}{2}$ and $\frac{n^2+2n}{2}$ of the coefficient matrix. Thus row $\frac{n^2+2n}{2}$ is a linear combination of the first n rows and row $\frac{n^2+2}{2}$. Similarly row $\frac{n^2+2n-2}{2}$ is a linear combination of rows $n+1$ to $2n$ and row $\frac{n^2+4}{2}$ of the coefficient matrix. Continuing this process we show that each of the rows i such that $\frac{n^2+n+2}{2} \leq i \leq \frac{n^2+2n}{2}$ is a linear combination of n rows of regularity conditions and row $n^2+n+1-i$. Therefore $\frac{n^2}{2}$ regularity conditions and the first $\frac{n}{2}$ row conditions are linearly independent. A similar argument shows that each of the last $\frac{n}{2}$ column conditions are linearly dependent on n regularity conditions and one of the remaining $\frac{n}{2}$ column conditions. Thus the rank of the coefficient matrix is $\frac{n^2}{2}+\frac{n}{2}+\frac{n}{2}=\frac{n^2+2n}{2}$. From the rank and nullity theorem we get the dimension of solution space to be equal to $n^2-\frac{n^2+2n}{2}=\frac{n(n-2)}{2}$. Hence the dimension of $\emptyset RM(n)$ is $\frac{n(n-2)}{2}$ when n is even. □

Theorem 2.4. *The dimension of a $\emptyset PM(n)$ is n^2-4n+3.*

Proof. Let $A=[a_{ij}]$ be a zero pandiagonal magic square. We create a system of homogeneous linear equations in n^2 variables as in the proof of Theorem 2.3. The system has $4n$ equations written in the order of n row conditions, n column conditions, n diagonal conditions parallel to the main diagonal and the n diagonal conditions parallel to the cross diagonal. When $n=5$ the 20-by-25 coefficient matrix appears as follows:

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

The first n rows of the coefficient matrix determined by A are clearly linearly independent. The sum of the first n rows of the coefficient matrix equals the sum of rows $n+1$ through $2n$. Thus row $2n$ can be written as a linear combination of the first n rows and rows $n+1$ through $2n-1$ and is therefore linearly dependent. The same can be said for row $3n$ which can be written as a linear combination of the first n rows and rows $2n+1$ through $3n-1$. Lastly row $4n$ can can be written as a linear combination of the first n rows and rows $3n+1$ through $4n-1$. It is easily checked that all rows of the coefficient matrix except rows $2n$, $3n$, and $4n$ form a linearly independent set. Thus the rank of the coefficient matrix determined by A is $4n-3$. By the rank and nullity theorem the dimension of the solution space is $n^2-(4n-3)$. Hence the dimension of $\emptyset PM(n)$ is n^2-4n+3. $\square$

We give below the basis for any $\emptyset RM(3)$, $\emptyset RM(4)$, and $\emptyset RM(5)$. A basis for $\emptyset RM(3)$ consists of the two matrices

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \text{and} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

A basis for $\emptyset RM(4)$ consists of the four matrices

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 2 \\ -2 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \text{and} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Using these basis vectors we can write a standard form for a magic square in $\emptyset RM(4)$ as

$$\begin{bmatrix} a & b & c & -a-b-c \\ d & -a-b-d & -a-c-d & 2a+b+c+d \\ -2a-b-c-d & a+c+d & a+b+d & -d \\ a+b+c & -c & -b & -a \end{bmatrix}$$

where a, b, c, d are complex numbers.

The standard form for a magic square in $\emptyset RM(5)$ is given below using a basis of eight matrices for the space

$$\begin{bmatrix} a & b & -(a+b+c+d) & c & d \\ e & f & -(e+f+g+h) & g & h \\ h+d-a-e & c+g-b-f & 0 & b+f-c-g & a+e-h-d \\ -h & -g & e+f+g+h & -f & -e \\ -d & -c & a+b+c+d & -b & -a \end{bmatrix}$$

where a, b, c, d, e, f, g, h are complex numbers.

Adding the n-by-n matrix E consisting of all 1's to a basis for $\emptyset RM(n)$ we get a basis for all n-by-n regular magic squares.

Corollary 2.5. *The dimension of the vector space of all n-by-n regular magic squares is $\frac{(n-1)^2}{2}+1$ when n is odd and $\frac{n(n-2)}{2}+1$ when n is even.*

A similar result for the vector space of all n-by-n pandiagonal matrix is as follows:

Corollary 2.6. *The dimension of the vector space of all n-by-n pandiagonal magic squares is n^2-4n+4.*

As a consequence of Corollary 2.6, the only pandiagonal matrix of size 3-by-3 is a constant multiple of E.

If we write a linear combination of the basis matrices of 3-by-3 regular matrices we obtain

$$\begin{bmatrix} a+b & a-b-c & a+c \\ a-b+c & a & a+b-c \\ a-c & a+b+c & a-b \end{bmatrix}.$$

This is a regular magic square with $\mu = 3a$. If we consider a 3-by-3 classical regular magic square then $\mu = 15$ and $a = 5$. If we select $b = -1$ and $c = -3$ we get the oldest known Lo-Shu magic square

$$\begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}.$$

3 Eigenvalues of a magic square

Since the field of complex numbers is algebraically closed, the characteristic polynomial $p_A(z)$ of an n-by-n semi-magic square A can be written as a product of linear facotrs,

$$p_A(z) = det(A + zI) = \sum_{i=1}^{n} (z - m_i)$$

where the complex numbers $m_1, m_2, ..., m_n$ are the eigenvalues of A. Since $Ae = \mu e$ where e is a n-by-1 column of 1's we observe that the semi-magic sum μ is an eigenvalue of A. We give a simpler proof to the following theorem found in [1].

Theorem 3.1. *If A is a semi-magic square and $p \in \mathbb{C}$, then $A + pE$, where E is the all 1's matrix, has the same eigenvalues $m_2, ..., m_n$ except that μ is replaced with $\mu + pn$.*

Proof. Since $AE = \mu E = EA$, there is a unitary matrix U such that $U^*AU = T$ and $U^*EU = S$ where T and S are upper triangular matrices with diagonal entries being the eigenvalues of A and E respectively. Since E has eigenvalues n of multiplicity one and 0 of multiplicity $n-1$, we may order them as $s_1, s_2, ..., s_n$. We observe that $A + pE$ is a semi-magic square with magic sum $\mu + pn$. Hence its eigenvalues are $\mu + pn, k_2, k_3, ...k_n$. Now $U^*(A + pE)U = T + pS$ has diagonal entries that are eigenvalues of $A + pE$. Thus

$$\mu + pn = \mu + ps_1,$$

$$k_i = m_i + ps_i \quad \text{for} \quad 2 \leq i \leq n$$

From the first equation we see that $s_1 = n$. Since $s_2, ..., s_n$ are all zero, it follows that $k_i = m_i$ for $2 \leq i \leq n$. □

Corollary 3.2. *If A is a semi-magic square, then $A - \frac{\mu}{n}E$ has the same eigenvalues $m_2, ..., m_n$ as A except that μ is replaced with 0.*

The above corollary is also proved in [10]. The following theorem and its proof are found in [7]

Theorem 3.3. *If $A = [a_{rs}]$ is a semi-magic square with nonnegative entries then*

$$|m_i| \leq \mu \quad \text{for } i = 2, ..., n$$

where μ, m_2,...,m_n are the eigenvalues of A.

Proof. Schur's theorem says that every square matrix A is unitarily equivalent to a triangular matrix $T = [t_{ij}]$ whose diagonal entries are the eigenvalues of A. Let $U = [u_{ij}]$ be the unitary matrix such that $U^*AU = T$. It follows from matrix multiplication that

$$t_{ij} = (U^*)_i[a_{rs}](U)_j = \sum_{r,s=1}^{n} \overline{u}_{ri} a_{rs} u_{sj}$$

Since entries of A are nonnegative and the columns of U are unit vectors we obtain

$$|t_{ij}| = |\sum_{r,s=1}^{n} a_{rs} \overline{u}_{ri} u_{si}| \leq \sum_{r,s=1}^{n} |a_{rs}||\overline{u}_{ri}||u_{si}|$$

$$\leq \frac{1}{2} \sum_{r,s} |a_{rs}|(|u_{ri}|^2 + |u_{si}|^2) = \frac{1}{2}[\sum_{r} |u_{ri}|^2 \sum_{s} a_{rs} + \sum_{s} |u_{si}|^2 \sum_{r} a_{rs}]$$

$$= \frac{1}{2}[\mu \sum_{r} |u_{ri}|^2 + \mu \sum |u_{si}|^2)] = \mu$$

□

Observation 3.4. *If A is a semi-magic square with positive entries, then from Perron's theorem [6, p500] it follows that μ is an eigenvalue with algebraic (hence, geometric) multiplicity one.*

4 Regular Magic Squares

Throughout section 4 we will assume that all matrices have real entries. Let J denote the permutation matrix obtained by writing 1 in each of the cross diagonal entries and 0 elsewhere. Since multiplying a matrix on the left by J reverses the order of the rows and multipling on the right by J reverses the columns, we will call J the *reversal* matrix. Also, observe that $J^T = J$ and $J^2 = I$. Thus J is its own inverse.

Using the reversal matrix J, the condition for regularity of a magic square can be written as

$$A + JAJ = (\frac{2\mu}{n})E \tag{1}$$

where E is the all 1's matrix.

Definition A n-by-n matrix B with real entries is said to be centrosymmetric if $JBJ = B$ and said to be centroskew if $JBJ = -B$.

Definition If A is a regular magic square we define

$$Z = A - \frac{\mu}{n}E$$

to be the *associated zero regular magic square.*

Lemma 4.1. *[10] If A is a regular magic square, the associated zero magic square is a centroskew matrix.*

Proof. Observe that using (1)

$$\begin{aligned} Z + JZJ &= (A - \tfrac{\mu}{n}E) + (JAJ - \tfrac{\mu}{n}E) \\ &= 0 \end{aligned}$$

□

This lemma allows us to use the known results on centroskew matricies found in [13], [5].

4.1 Odd Order Sqaures

In the following theorem, we give a necessary and sufficient condition for an odd order regular magic square to be nonsingular.

Theorem 4.2. *Let A be a regular magic square of order $n = 2k+1$ with positive entries. Suppose the associated centroskew matrix Z is written in partitioned form as follows:*

$$Z = \begin{bmatrix} Z_{11} & a & Z_{13} \\ b^T & 0 & -b^T J \\ -JZ_{13}J & -Ja & -JZ_{11}J \end{bmatrix}$$

where Z_{11}, Z_{13} are k-by-k matrices and a, b are k-by-1 vectors. Then A is nonsingular if and only if $Z_{11} + Z_{13}J$ and $Z_{11} - Z_{13}J$ are both nonsingular.

Proof. Using partitioned matrices

$$K = \begin{bmatrix} I & 0 & I \\ 0 & 1 & 0 \\ J & 0 & -J \end{bmatrix} \quad \text{and} \quad K^{-1} = \frac{1}{2}\begin{bmatrix} I & 0 & J \\ 0 & 2 & 0 \\ I & 0 & -J \end{bmatrix}$$

where I, J are $k \times k$ matrices we find Z' similar to Z, namely

$$Z' = K^{-1}ZK = \begin{bmatrix} 0 & 0 & Z_{11} - Z_{13}J \\ 0 & 0 & 2b^T \\ Z_{11} + Z_{13}J & a & 0 \end{bmatrix}.$$

We write $Z' - \lambda I$ in partitioned form as

$$Z' - \lambda I = \begin{bmatrix} -\lambda I & C_{12} \\ C_{21} & -\lambda I \end{bmatrix}$$

where $C_{12} = \begin{bmatrix} Z_{11} - Z_{13}J \\ 2b^T \end{bmatrix}$ and $C_{21} = \begin{bmatrix} Z_{11} + Z_{13}J & 2a \end{bmatrix}$. Note that C_{12} is a $(k+1)$-by-k matrix and C_{21} is a k-by-$(k+1)$ matrix. Using elmtentary row operations we reduce $Z' - \lambda I$ to

$$Z' - \lambda I \sim \begin{bmatrix} -\lambda I & C_{12} \\ 0 & \frac{1}{\lambda}C_{21}C_{12} - \lambda I \end{bmatrix}.$$

Therefore we can write the characteristic polynomial of Z' as

$$\begin{aligned} \det\left(Z' - \lambda I\right) &= (-1)^{k+1}\lambda^{k+1}\det\left(\tfrac{1}{\lambda}C_{21}C_{12} - \lambda I\right) \\ &= (-1)^{k+1}\lambda\det\left(C_{21}C_{12} - \lambda^2 I\right) \end{aligned}$$

Since Z is a zero regular magic square we observe that

$$a = -(Z_{11} + Z_{13})e \quad \text{and} \quad b^T = e^T(JZ_{13}J - Z_{11})$$

Thus

$$\begin{aligned} C_{21}C_{12} &= \begin{bmatrix} Z_{11} + Z_{13}J & 2a \end{bmatrix}\begin{bmatrix} Z_{11} - Z_{13}J \\ 2b^T \end{bmatrix} \\ &= (Z_{11} + Z_{13}J)(Z_{11} - Z_{13}J) + (Z_{11} + Z_{13})2E(Z_{11} - JZ_{13}J) \\ &= (Z_{11} + Z_{13}J)(I + 2E)(Z_{11} - Z_{13}J) \end{aligned}$$

Note that $\det\left(I + 2E\right) = 2k + 1$ where I and E are k-by-k matrices. To see this, observe that adding columns 2 through k to the first column gives a constant in the first column consisting of $2k + 1$. Then a row reduction produces an upper triangular matrix with $(2k + 1)$ in $(1, 1)$ entry and 1 for the remaining $k - 1$ diagonal entries.

Hence λ^2 is a factor of $\det\left(C_{21}C_{12} - \lambda^2 I\right)$ if $\det\left(C_{21}C_{12}\right) = 0$. Using Corollary 3.2 the eigenvalues of Z are the same as A except 0 replaces the eigenvalues μ of A. Therefore zero is an eigenvalue of an odd order regular magic square M if and only if $Z_{11} + Z_{13}J$ and $Z_{11} - Z_{13}J$ are nonsingular. $\square$

4.2 Even Order Squares

Let us assume A is a regular magic square of even order. If $Z = A - \frac{\mu}{n}E$ is of order $2k$ then we partition Z into $k \times k$ block matrices

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ -JZ_{12}J & -JZ_{11}J \end{bmatrix}$$

Using invertible matrices

$$K = \begin{bmatrix} I & I \\ J & -J \end{bmatrix} \quad \text{and} \quad K^{-1} = \frac{1}{2}\begin{bmatrix} I & J \\ I & -J \end{bmatrix}$$

we find a matrix Z' similar to Z, namely

$$Z' = K^{-1}ZK = \begin{bmatrix} 0 & Z_{11} + Z_{12}J \\ Z_{12} - Z_{12}J & 0 \end{bmatrix}$$

where each block is size k-by-k. Using elementary row operations we reduce

$$Z' - \lambda I = \begin{bmatrix} -\lambda I & Z_{11} + Z_{12}J \\ Z_{11} - Z_{12}J & -\lambda I \end{bmatrix}$$

to an upper triangular block matrix

$$Z' - \lambda I = \begin{bmatrix} -\lambda I & Z_{11} + Z_{12}J \\ 0 & \frac{1}{\lambda}(Z_{11} - Z_{12}J)(Z_{11} + Z_{12}J) - \lambda I \end{bmatrix}.$$

Hence

$$\begin{aligned} \det\big(Z' - \lambda I\big) &= (-1)^k \lambda^k \det\big(\tfrac{1}{\lambda}(Z_{11} - Z_{12}J)(Z_{11} + Z_{12}J) - \lambda I\big) \\ &= (-1)^k \det\big((Z_{11} - Z_{12}J)(Z_{11} + Z_{12}J) - \lambda^2 I\big) \end{aligned} \tag{2}$$

From this we obtain the following result first proved by [10] using different methods.

Theorem 4.3. *Let A be an n-by-n regular magic square with positive entries. If n even, then A is singular.*

Proof. By Corollary 3.2 the eigenvalues of $Z = A - \frac{\mu}{n}E$ are the same as the eigenvalues of A except that 0 replaces μ. Since the eigenvalues of Z and Z' are the same, we consider (2) from above. If $\det(Z_{11} - Z_{12}J)(Z_{11} + Z_{12}J)$ is zero then 0 is an eigenvalue of Z of multiplicity at least two. Since Z is a zero magic square the columns of $Z_{11} + Z_{12}J$ sum to zero and hence $\det\big(Z_{11} + Z_{12}J\big) = 0$. This implies λ^2 is a factor in the characteristic polynomial of Z. Therefore, A must have eigenvalue 0 when n is even. Hence, A is singular. □

4.3 Singular Magic Squares

Unlike the even case, the odd order regular magic sqaures may be singular or nonsingular. The Lo-Shu magic square mentioned in Section 3 is nonsingular. In [8] , using computer experimentation, the authors have found 656 singular regular magic squares of order 5. As seen from the proof of Theorem 4.2, given an odd order regular magic sqaure the associated centroskew zero magic square can have 0 as an eigenvalue with odd multiplicites. Therefore, any odd order regular magic square may have 0 as an eigenvalue with even multilplicities. The following example of regular magic square of order 5 with characteristic polynomial $P_A(z) = z^4(z-65)$ is given in [8]:

$$\begin{bmatrix} 2 & 11 & 21 & 23 & 8 \\ 16 & 14 & 7 & 6 & 22 \\ 25 & 17 & 13 & 9 & 1 \\ 4 & 20 & 19 & 12 & 10 \\ 18 & 3 & 5 & 15 & 24 \end{bmatrix}$$

In the remaining part of this paper we present two methods of construction, one producing singular regular magic squares and the other producing nonsingular regular magic squares of odd order.

5 Composite Magic Squares

In this section we describe a construction of magic squares that are always singular.

Definition Let A be a m-by-n matrix and B be a p-by-q matrix. Then the Kronecker product (or tensor product) of A and B is defined as the mp-by-nq matrix

$$A_{n\times m} \otimes B_{p\times q} = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{mn}B \end{bmatrix}.$$

Suppose A is an n-by-n magic square with entries 1 through n^2 and B is a m-by-m magic square with entries 0 through m^2-1. Define

$$P = (E_m \otimes A) + (n^2 B \otimes E_n) \tag{3}$$

where E_m and E_n are all 1's matrices of size m-by-m and n-by-n respectively.

Lemma 5.1. *P defined in* (3) *is a classical magic square of order mn.*

Proof. Consider $Q = E_m \otimes A$. This is a mn-by-mn matrix written in the m-by-m block form as

$$Q = \begin{bmatrix} A & A & \dots & A \\ A & A & \dots & A \\ \vdots & \vdots & \ddots & \vdots \\ A & A & \dots & A \end{bmatrix}.$$

Now, consider $R = n^2 B \otimes E_n$. This is a mn-by-mn matrix of written in the m-by-m block form as

$$R = \begin{bmatrix} n^2 b_{11}E_n & n^2 b_{12}E_n & \dots & n^2 b_{1m}E_n \\ n^2 b_{21}E_n & n^2 b_{22}E_n & \dots & n^2 b_{2m}E_n \\ \vdots & \vdots & \ddots & \vdots \\ n^2 b_{m1}E_n & n^2 b_{m2}E_n & \dots & n^2 b_{mm}E_n \end{bmatrix}.$$

Using block addition P can be written in the m-by-m block form as

$$P = \begin{bmatrix} A + n^2 b_{11}E_n & A + n^2 b_{12}E_n & \dots & A + n^2 b_{1m}E_n \\ A + n^2 b_{21}E_n & A + n^2 b_{22}E_n & \dots & A + n^2 b_{2m}E_n \\ \vdots & \vdots & \ddots & \vdots \\ A + n^2 b_{m1}E_n & A + n^2 b_{m2}E_n & \dots & A + n^2 b_{mm}E_n \end{bmatrix}.$$

Since the entries of A are 1 through n^2 and $b_{ij} \in \{0, 1, 2, \dots, m^2 - 1\}$ we observe that P has entries 1 through m^2n^2.

If A has magic sum μ and B has magic sum ρ, it follows that Q is a magic square with magic sum $m\mu$ and R is a magic square with magic sum $n^3\rho$. Since P is a sum of Q and R it is a magic square with magic sum $m\mu + n^3\rho$. □

Lemma 5.2. *If A and B are regular magic squares, then P defined in 3 is a regular magic square.*

Proof. Consider $P = Q + R$ where Q and R are defined in the proof of Lemma 5.1. Now $JPJ = JQJ + JRJ$ where

$$JQJ = \begin{bmatrix} JAJ & JAJ & \dots & JAJ \\ JAJ & JAJ & \dots & JAJ \\ \vdots & \vdots & \ddots & \vdots \\ JAJ & JAJ & \dots & JAJ \end{bmatrix}$$

and

$$JRJ = \begin{bmatrix} n^2b_{mm}E_n & n^2b_{m(m-1)}E_n & \dots & n^2b_{m1}E_n \\ n^2b_{(m-1)m}E_n & n^2b_{(m-1)(m-1)}E_n & \dots & n^2b_{(m-1)1}E_n \\ \vdots & \vdots & \ddots & \vdots \\ n^2b_{1m}E_n & n^2b_{1(m-1)}E_n & \dots & n^2b_{11}E_n \end{bmatrix}$$

Therefore

$$\begin{aligned} P + JPJ &= (Q + JQJ) + (R + JRJ) \\ &= \tfrac{2\mu}{n}E_{mn} + n^2\tfrac{2\rho}{m}E_{mn} \\ &= \tfrac{2(m\mu+n^3\rho)}{mn}E_{mn}, \end{aligned}$$

proving that P is regular. □

In [4], the same construction called "compouding of magic squares" is discussed and Lemma 5.2 is proved. The following result is also proved in [4].

Lemma 5.3. *If A and B are pandiagonal magic squares, then P defined in 3 is a pandiagonal magic square.*

In addition to proving Lemma 5.3, we have the following result showing P is singular.

Theorem 5.4. *Suppose A is an n-by-n magic square with entries 1 through n^2, B is a m-by-m magic square with entries 0 through $m^2 - 1$, E_n and E_m are all 1's matrices of size n-by-n and m-by-m respectively. Then*

$$P = (E_m \otimes A) + (n^2B \otimes E_n)$$

is a singular classical magic square of order mn.

Proof. It is enough to show that P is not of full rank. Since rank of a sum of two matrices is never more than the sum of their individual ranks we get

$$rank(P) \leq rank(Q = E_m \otimes A) + rank(n^2B \otimes E_n).$$

Since rank of a tensor product of two matrices is equal to the product of their ranks, we get

$$rank(P) \leq rank(E_n)rank(A) + rank(B)rank(E_n).$$

Since the all 1's matrix has rank 1, and the full ranks of A and B are respectively n and m, we get

$$rank(P) \leq n + m$$

Without loss of generality, let $m > 2$ and $n > 2$. Then $rank(P) \leq n + m < mn$. Thus P is singular. □

6 A Construction of Regular Magic Squares that are Nonsingular

In this section we describe a method to construct nonsingular regular magic squares of prime order using circulant matrices and orthogonal latin squares.

Definition A *latin square* is an n-by-n matrix containing n distinct numbers arranged in such a way that each number appears exactly once in every row and in every column.
For further information on latin squares see [3].

Definition Two n-by-n latin squares A and B are said to be *orthogonal* if the n^2 ordered pairs obtained using the corresponding entries of A and B are disticnt. In this case, each matrix is said to be an *orthogonal mate* of the other.

Definition A circulant matrix is a matrix of order n in which each row other than the first row is obtained from the preceding row by shifting entries cyclically one column to the right.

Here is an example of a 5-by-5 circulant matrix:

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_5 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_1 \end{bmatrix}$$

.

Observation 6.1. *Suppose A is a circulant matrix of odd order with n distinct entries in the first row. Then A satisfies the following conditions:*

- *A has constant main diagonal and constant broken diagonals parallel to the main diagonal (i.e., A is a Toeplitz matrix)*
- *the cross diagonal and all broken diagonals parallel to the cross diagonal of A have n distinct entries*
- *A is a latin square*

Let P be the right shifting permutation matrix with all ones on the superdiagonal and one in the lower left corner. For example, in the 5-by-5 case it is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that if a matrix A is multiplied by P on the right, then each column of A is shifted to its right and the last column is shifted to the first column of A. Also, observe that P^n = I and that the eigenvalues of P are the n^{th} roots of unity [9].

For the rest of the section let n *denote an odd integer* $2k+1$ *and let* S *denote the set of numbers* $\{-k, -k+1, ..., -1, 0, 1, ..., k-1, k\}$. Let A be a circulant matrix of order n whose first row $(a_1, a_2, ..., a_n)$ consists of n distinct members from S. Such a circulant matrix is said to be a S-circulant matrix.

For example

$$\begin{bmatrix} 0 & -1 & -2 & 1 & 2 \\ 2 & 0 & -1 & -2 & 1 \\ 1 & 2 & 0 & -1 & -2 \\ -2 & 1 & 2 & 0 & -1 \\ -1 & -2 & 1 & 2 & 0 \end{bmatrix}$$

is a s-circulant matrix of order 5.

Lemma 6.2. *Suppose* A *is a S-circulant matrix with* $a_1 = 0$. *Then* A *is a zero magic square.*

Proof. Since the sum of the members of S is zero, using Observation 6.1 we get the result. ☐

Lemma 6.3. *Suppose* A *is a S-circulant matrix with* $a_1 = 0$. *If the remaining entries of the first row satisfy the condition that*

$$a_j + a_{n+2-j} = 0, \qquad j = 2, ..., n \tag{4}$$

then A *is a centroskew matrix.*

Proof. Let A be a S-circulant matrix with $a_1 = 0$. Then A can be written as $A = \sum_{j=0}^{n-1} a_{j+1} P^j + a_1 J$ where P is the right shifting permutation matrix. Since JAJ $= \sum_{j=1}^{n-1} a_{n+1-j} P^j$ we get A+JAJ $= \sum_{j=1}^{n-1} (a_{j+1} + a_{n+1-j}) P^j = 0$ using $a_1 = 0$ and the assumption (4). Hence A is centroskew. ☐

Example of a centroskew S-circulant matrix with $a_1 = 0$.

$$A = \begin{bmatrix} 0 & 2 & -1 & 1 & -2 \\ -2 & 0 & 2 & -1 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ -1 & 1 & -2 & 0 & 2 \\ 2 & -1 & 1 & -2 & 0 \end{bmatrix} \tag{5}$$

Lemma 6.4. *Let* A *be a S-circulant matrix with* $a_1 = 0$ *that is centroskew. Then* JA *and* AJ *are the orthogonal mates of* A *where* J *is the reversal matrix.*

Proof. We will show that A and JA are orthogonal mates. The proof of the other case is similar. Multiplication of A by J on the right reverses the order of the columns of A. Therefore the (broken) diagonals parallel to the main diagonal interchange to (broken) diagonals parallel to the cross diagonal. Hence, if we pair up entries of A and AJ using Observation 6.1, each constant diagonal pairs up with distinct entries from S. Therefore A and AJ are orthogonal latin squares. ☐

Lemma 6.5. *Let* A *be a S-circulant matrix with* $a_1 = 0$ *that is centroskew. Then* AJ *and* JA *are centroskew.*

Proof. Note that $J(JA)J = J(JAJ) = -JA$. Hence JA is centroskew. Simiarly AJ is centroskew. ☐

Theorem 6.6. *Let A be a S-circulant matrix of order n with $a_1 = 0$ that is centroskew. Define*

$$Z = nA + AJ \tag{6}$$

Then,

1. *Z is centroskew*
2. *Z is a zero magic square with n^2 distinct entries from the set $\{-(\frac{n^2-1}{2}), ..., -1, 0, 1, ..., (\frac{n^2-1}{2})\}$*

Proof. Since $JZJ = nJAJ + J(AJ)J = -(nA + AJ) = -Z$ we see that Z is centroskew. Since A has distinct entries from the set S, nA has distinct entries from the set $T = \{-n(\frac{n^2-1}{2}), -n(\frac{n^2-3}{2}), ..., -n, 0, n, ..., n(\frac{n^2-3}{2}), n(\frac{n^2-1}{2})\}$. The entries of AJ are distinct elements from $S = \{-(\frac{n^2-1}{2}), ..., -1, 0, 1, ..., (\frac{n^2-1}{2})\}$. From Lemma 6.4 we know that A and AJ are orthogonal latin squares. Hence nA and AJ are orthogonal latin squares. Since entries of Z are sums of n^2 distinct ordered pairs of entries from T and S (in that order). Z has n^2 distinct entries ranging from $-(\frac{n^2-1}{2})$ through $(\frac{n^2-1}{2})$. Since nA and AJ are zero magic squares Z is a zero magic square with n^2 distinct entries from the set $\{-(\frac{n^2-1}{2}), ..., -1, 0, 1, ..., (\frac{n^2-1}{2})\}$. ☐

Theorem 6.7. *Let Z be as defined in (6). Then rank $Z = n - 1$ when n is an odd prime number.*

Proof. Let Z = nA+AJ = A(nI+J) where A is a S-circulant centroskew matrix with $a_1 = 0$. It is easily observed that columns of nI+J are linearly independent. Hence rank(nI+J) = n, the order of the matrix. Hence rank Z = rank A.

Since A is a S-circulant matrix its eigenvalues are determined by its first row [9] by the formula

$$\{\sum_{j=0}^{n-1} a_{j+1} w^{kj} : k = 0, 1, ..., n-1\}$$

where $w := e^{\frac{2\pi i}{n}}$.

For k = 0 we find that the eigenvalue is zero. Since n is a prime number, $0 \leq k, j \leq n-1$ are all relatively prime to n. Hence for each value of $k = 1, ..., n-1$ kj will have distinct values mod n. Since $a_1 = 0$ and $a_2, ..., a_n$ are form S we find that the remaining eigenvalues are nonzero. Therefore rank $A = n - 1$. This proves the theorem. ☐

Theorem 6.8. *Let A be a s-circulant matrix with $a_1 = 0$ that is also centroskew. Suppose n is an odd prime and $Z = nA + AJ$. Then $M = Z + \frac{n^2+1}{2}E$ is a classical regular magic square that is nonsingular.*

Proof. Since $Z = nA + AJ$ is a zero regular magic square with n^2 distinct entries ranging from $-(\frac{n^2-1}{2})$ to $(\frac{n^2-1}{2})$ (from Theorem 6.6) we see that M is a classical regular magic square. Since Z is a centroskew matrix associated with M that has zero as an eigenvalue with multiplicity one we conclude from Corollary 3.2 that M is nonsingular. ☐

Using Theorem 6.8 and the matrix A given in 5 we get the following 5-by-5 classical regular magic square that is nonsingular:

$$\begin{bmatrix} 11 & 24 & 7 & 20 & 3 \\ 4 & 12 & 25 & 8 & 16 \\ 17 & 5 & 13 & 21 & 9 \\ 10 & 18 & 1 & 14 & 22 \\ 23 & 6 & 19 & 2 & 15 \end{bmatrix}.$$

7 Conclusion

This research stemmed from Bruce Mattingly's proof that even order regular magic squares are singular [10]. We were interested in finding results for odd order regular magic squares. We have been able to formulate methods of construction for a number of different magic sqaures with different properties. Using the method described in section 6, we found it possible to construct nonsingular odd order magic squares that are not prime numbers. The following is a 15-by-15 regular magic square that is nonsingular.

$$\begin{bmatrix}
218 & 91 & 204 & 77 & 190 & 63 & 176 & 49 & 162 & 35 & 148 & 21 & 134 & 7 & 120 \\
105 & 203 & 76 & 189 & 62 & 175 & 48 & 161 & 34 & 147 & 20 & 133 & 6 & 119 & 217 \\
202 & 90 & 188 & 61 & 174 & 47 & 160 & 33 & 146 & 19 & 132 & 5 & 118 & 216 & 104 \\
89 & 187 & 75 & 173 & 46 & 159 & 32 & 145 & 18 & 131 & 4 & 117 & 215 & 103 & 201 \\
186 & 74 & 172 & 60 & 158 & 31 & 144 & 17 & 130 & 3 & 116 & 214 & 102 & 200 & 88 \\
73 & 171 & 59 & 157 & 45 & 143 & 16 & 129 & 2 & 115 & 213 & 101 & 199 & 87 & 185 \\
170 & 58 & 156 & 44 & 142 & 30 & 128 & 1 & 114 & 212 & 100 & 198 & 86 & 184 & 72 \\
57 & 155 & 43 & 141 & 29 & 127 & 15 & 113 & 211 & 99 & 197 & 85 & 183 & 71 & 169 \\
154 & 42 & 140 & 28 & 126 & 14 & 112 & 225 & 98 & 196 & 84 & 182 & 70 & 168 & 56 \\
41 & 139 & 27 & 125 & 13 & 111 & 224 & 97 & 210 & 83 & 181 & 69 & 167 & 55 & 153 \\
138 & 26 & 124 & 12 & 110 & 223 & 96 & 209 & 82 & 195 & 68 & 166 & 54 & 152 & 40 \\
25 & 123 & 11 & 109 & 222 & 95 & 208 & 81 & 194 & 67 & 180 & 53 & 151 & 39 & 137 \\
122 & 10 & 108 & 221 & 94 & 207 & 80 & 193 & 66 & 179 & 52 & 165 & 38 & 136 & 24 \\
9 & 107 & 220 & 93 & 206 & 79 & 192 & 65 & 178 & 51 & 164 & 37 & 150 & 23 & 121 \\
106 & 219 & 92 & 205 & 78 & 191 & 64 & 177 & 50 & 163 & 36 & 149 & 22 & 135 & 8
\end{bmatrix}$$

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