Distance-Regular Cayley Graphs

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Abstract

This paper is a survey of distance-regular Cayley graphs of diameter two and three, primarily focusing on diameter three (due to the extensive amount of research already done in diameter two). The graphs were classified by their properties and related eigenvalues and multiplicities. Each classification was studied in depth to determine whether certain patterns held. Though many of the graphs could be formed in abelian groups, some were found to be genuinely non-abelian. Of special interest were the graphs formed by the dihedral and cyclic groups. In addition the existence of a (400,21,2,1) strongly regular graph is explored.

Contents

1 Preliminaries **3**
1.1 Definitions 3

2 Tools Used in Analyzing the Graphs **5**
2.1 Intersection Numbers 5
2.2 Matrices 6

3 Classifications **9**
3.1 Classifications of the graphs 9
3.2 Algebraic Classifications 10
3.3 Multiplicities 10

4 Findings and Theorems **11**
4.1 The graph generated by $\mathbb{Z}_{2p}$ 11
4.2 Taylor and Bipartite graphs 13
4.3 Dihedral groups 15

5 An Interesting Graph **17**
5.1 Definitions 17
5.2 Proof 17

6 Appendix **19**

Bibliography
20

1 Preliminaries

We will begin with the basic definitions of distance-regular graphs and Cayley graphs. Distance-regular graphs have very specific properties which are interesting and we will focus primarily on diameter 3 graphs. Cayley graphs are also defined. Cayley graphs are interesting because they allow algebra to be employed to study their properties since they obey the laws of groups. Our definitions come from [1] and [3]

1.1 Definitions

Definition 1 (Distance-Regular Graph) *Given two vertices x and y in a graph of distance h define*

$$P^h_{i,j} = |\{z | z \in \text{the graph}, \partial(x,z) = i, \partial(y,z) = j\}\ |$$

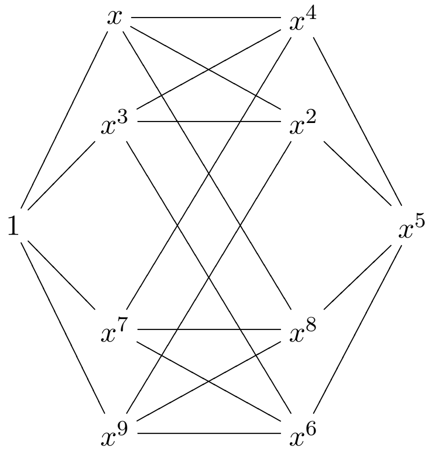
A graph is said to be distance-regular if $P^h_{i,j}(x,y)$ does not depend on x and y, but only h.

The diameter of the graph $(d) = \max\ \{\ \partial(x,y)|\ x, y \in \mathrm{V(G)}\}$. Throughout this paper we will refer to v as the number of vertices in a graph and k as the degree of the vertices.

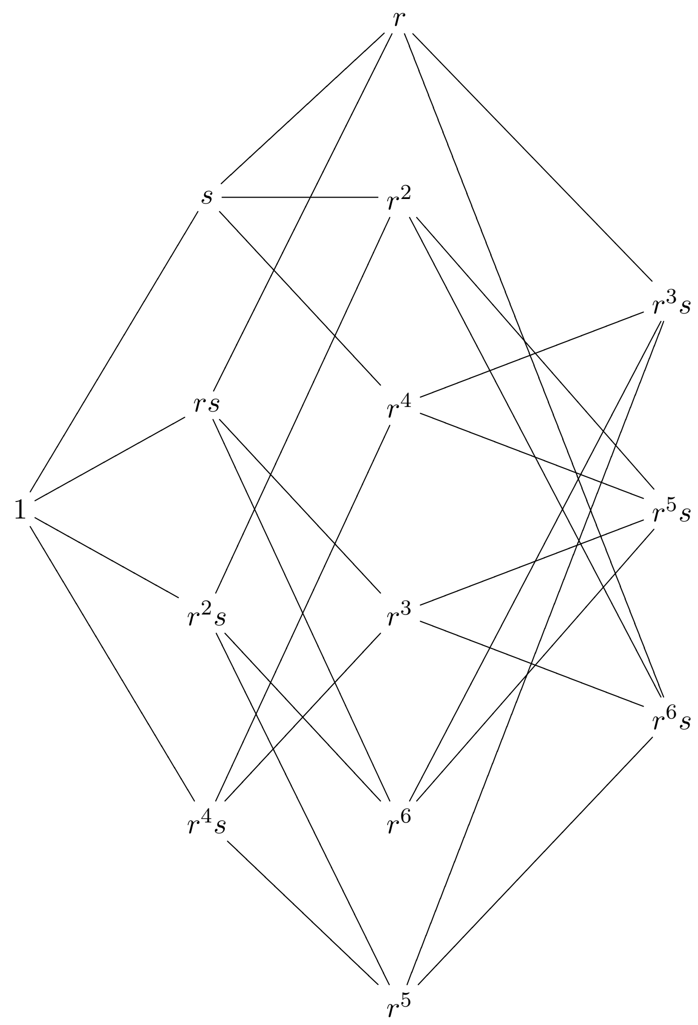
Definition 2 (Cayley Graph) *A Cayley graph is a graph whose vertices are all elements of a group. The graph can be generated by a subset of that group, which is called S. For all vertices x and y in the graph, x is adjacent to y if and only if $xs = y$ for some $s \in S$*

In this paper we are dealing with only undirected graphs, so for every $x \in S$, $x^{-1} \in S$.

Below is an example of both a Cayley and distance-regular graph. It is $\mathbb{Z}_{10}$ and S= $\{x, x^3, x^7, x^9\}$ and degree 4.



Another example is the graph generated by the Dihedral group of order 14 and degree 4. $S = \{s,\, rs,\, r^2s,\, r^4s\}$



Eigenvalues: -$\sqrt{3}$, $\sqrt{3}$, -4, 4
Multiplicities: 6, 6, 1, 1

2 Tools Used in Analyzing the Graphs

In order to study distance-regular graphs, we will use several tools to find important properties, eigenvalues and the eigenvalues' respective multiplicity. Our notation follows that of [1] and [6].

2.1 Intersection Numbers

Definition 3 (Intersection Numbers) *The intersection numbers result directly from the values of $P^h_{i,j}$. We define the intersection numbers as follows:*

$$\begin{aligned} c_i &= P^i_{1,i-1} \text{ for } (1 \leq i \leq d) \\ a_i &= P^i_{1,i} \text{ for } (0 \leq i \leq d) \\ b_i &= P^i_{1,i+1} \text{ for } (0 \leq i \leq d\text{-}1) \end{aligned}$$

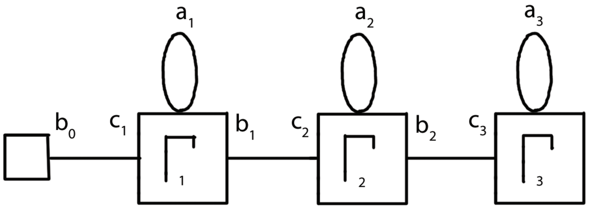
A convenient way to find and display the intersection numbers is the diagram below. This diagram can begin with any single vertex. The graph is divided into adjacency classes Γ_1, Γ_2, and Γ_3. If x is are starting vertex, these are defined as follows:

$$\Gamma_1(x) := \{y|\partial(x,y) = 1\}$$

$$\Gamma_2(x) := \{y|\partial(x,y) = 2\}$$

$$\Gamma_3(x) := \{y|\partial(x,y) = 3\}$$

which include all the vertices distance 1, 2, or 3 away from the starting vertex respectively. c_j is the number of ways a vertex in Γ_j can get 1 step closer to the starting vertex. Any vertex in Γ_j is adjacent to a_j vertices in Γ_j. b_j is the number of ways a vertex in Γ_j can get 1 step closer to the starting vertex.



Definition 4 (Intersection Array) *The intersection array is particular way of arranging the intersection numbers:*

$$\begin{array}{c|cccccc|} c & * & c_1 & c_2 & \dots & c_{d-1} & c_d \\ a & 0 & a_1 & a_2 & \dots & a_{d-1} & a_d \\ b & b_0 & b_1 & b_2 & \dots & b_{d-1} & * \end{array}$$

There are additional conditions that must hold with the intersection numbers. (from [1] pg. 158, Proposition 20.4)

$$|\Gamma_{j-1}|b_{j-1} = |\Gamma_j|c_j \ (1 \leq j \leq d)$$
$$1 \leq c_2 \leq c_3 \leq \ldots \leq c_d.$$
$$\text{k} \geq b_1 \geq b_2 \geq \ldots \geq b_{d-1}.$$
$$a_i + b_1 + c_1 = k$$

Due to the last property listed, there is a simpler way to write the intersection array.

$$\{ \ b_0, \ \ b_1, \ \ \ldots \ \ b_{d-1}; \ \ c_1, \ \ c_2, \ \ \ldots \ \ c_d \ \}$$

Note: This form of the intersection array does not include c_0 or b_d since these are not defined.

2.2 Matrices

Matrices are crucial in studying distance-regular graphs as it allows us to find eigenvalues and multiplicities. The eigenvalues and multiplicities carry important information which we will examine. We will also see how to create a simplified matrix that contains all the necessary information on eigenvalues and multiplicities.

Definition 5 (Adjacency Matrix) *The adjacency matrix (A_l) is a $v \times v$ matrix with entries $a_{i,j}$ where $\forall$ u_i and u_j in the graph*

$$a_{ij} = \begin{cases} 1 & \text{if } \partial(u_i, u_j) = l \\ 0 & \text{if } \partial(u_i, u_j) \neq l \end{cases}$$

In general, A_1 is denoted simply A.

With a matrix we can then find eigenvalues for the graphs which can be used in studying the graphs. We can find eigenvalues θ_1, θ_2, θ_3 and θ_4 as there will always be $d + 1$ distinct eigenvalues of the graph (from [5] pg. 566, F61.). Each of these eigenvalues also has a respective multiplicity. The eigenvectors form a subspace which is called the eigenspace. The dimension of the eigenspace is the multiplicity. Another way to view this is if the $v \times v$ adjacency matrix is diagonalized, the multiplicity of an eigenvalue is the number of times the eigenvalue appears on the diagonal. Note: We know that k will always be an eigenvalue with multiplicity 1 (from [1] Proposition 3.1)

Definition 6 (Intersection Matrix) *The intersection matrix is a useful tool to help us analyze distance-regular graphs. The matrix has the following* *arrangement:*

$$\mathbf{B} = \begin{pmatrix} a_0 & b_0 & & & & \boldsymbol{0} \\ c_1 & a_1 & b_1 & & & \\ & c_2 & a_2 & b_2 & & \\ & & c_3 & a_3 & \ddots & \\ & & & \ddots & \ddots & b_{d-1} \\ \boldsymbol{0} & & & & c_d & a_d \end{pmatrix}$$

This matrix is a simpler way to find all the information that we need from the larger adjacency matrix A.

Theorem The intersection matrix (B) has the same eigenvalues as the adjacency matrix (A).

Proof: (This is shown for diameter 3, however it can be generalized for that all distance-regular graphs [1])
Let λ be an eigenvalue of B. Then $\exists$ an eigenvector v such that B$\vec{v} = \lambda\vec{v}$. Let $\vec{v}$ =

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

Thus by the relationship of eigenvalues and eigenvector, B$\vec{v} = \lambda\ \vec{v}$. Using matrix multiplication we get

$$\lambda v_1 = kv_2,$$

$$\lambda v_2 = v_1 + a_1v_2 = b_2v_3,$$

$$\lambda v_3 = c_2v_2 + a_2v_3 + b_3v_4,$$

$$\lambda v_4 = c_3v_3$$

If we arrange A according to its associate classes as shown below, we can create an eigenvector by extending $\vec{v}$ to $\vec{v}*$. This is done by including $|\Gamma_{i-1}|$ copies of each v_i.

$$\begin{array}{c} \\ u_0 \\ u_1 \\ \vdots \\ u_k \\ u_{k+1} \\ \vdots \\ u_m \\ u_{m+1} \\ \vdots \\ u_v \end{array} \left(\begin{array}{c|ccc|ccc|ccc} u_0 & u_1 & \dots & u_k & u_{k+1} & \dots & u_m & u_{m+1} & \dots & u_v \\ \hline 0 & 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ \hline 1 & & & & & & & & & \\ \vdots & & \mathbf{a_1} & & & \mathbf{b_1} & & & \mathbf{0} & \\ 1 & & & & & & & & & \\ \hline 0 & & & & & & & & & \\ \vdots & & \mathbf{c_2} & & & \mathbf{a_2} & & & \mathbf{b_2} & \\ 0 & & & & & & & & & \\ \hline 0 & & & & & & & & & \\ \vdots & & \mathbf{0} & & & \mathbf{c_3} & & & \mathbf{a_3} & \\ 0 & & & & & & & & & \end{array}\right)$$

In the matrix above we have that the row and column sum of the entries in the matrix $\mathbf{a_i}$ equals a_i. Similarly, the row and column sum of the entries in the matrix $\mathbf{b_i}$ equals b_i. The same idea applies to $\mathbf{c_i}$ and all the entries in $\mathbf{0}$ are 0.

By using matrix multiplication, A$\vec{v}*$ gives us the same equations as B$\vec{v}$. Thus λ is the same for both matrices. Thus A always gives us at least 4 eigenvalues, all of which are shared by the intersection matrix B.

Now we show that A has exactly 4 eigenvalues.

Using the properties of the graph we get AA_i= $b_{i-1}A_{i-1}$ + a_iA_i + $c_{i+1}A_{i+1}$

$$AA_1 = A^2 = b_0 i + a_1 A + c_2 A_2 \tag{1}$$

$$AA_2 = b_1 A + a_2 A_2 + c_3 A_3 \tag{2}$$

$$AA_3 = b_2 A_2 + a_3 A_3 \tag{3}$$

By substitution, we get I, A, A_2, A_3 in terms of I, A, A^2, A^3 and vice versa. Furthermore, span {I, A, A_2, A_3 } = span {I, A, A^2, A^3}. So A^4 can be written as a linear combination of {I, A, A^2, A^3} in the form A^4 = $\alpha_1 A^3$ + $\alpha_2 A^2$ + $\alpha_3 A$ + $\alpha_4 I$. Therefore, $\exists$ P(A) = 0, where P(A) = A^4 - ($\alpha_1 A^3$ + $\alpha_2 A^2$ + $\alpha_3 A$ + $\alpha_4 I$) = 0. If we multiply both sides by an eigenvector $\vec{v}*$, we get P(A)$\vec{v}*$ = 0$\vec{v}*$. This allows A to be replaced by λ. Therefore we get $\alpha_1\lambda^3$ - $\alpha_2\lambda^2$ - $\alpha_3\lambda$ - 1, which equals 0 since 0$\vec{v}*$ = 0. Thus we have a fourth root polynomial which has at most 4 roots, which are the eigenvalues. Thus B and A have the same eigenvalues.

We get equations that describe our multiplicities by the trace of certain matrices. We can diagonalize A by Q. We get D=QAQ^{-1}.
Tr(D)=Tr(QAQ^{-1})
A property of linear algebra is Tr(AB)=Tr(BA)
So, Tr((QA)Q^{-1})=Tr(Q^{-1}(QA))
So, Tr(D)=Tr(A)
We can then create equations from this.
Tr(I)=V gives us

$$1 + f + g + h = V$$

Tr(D)=Tr(A) gives us

$$0 = k + \lambda_1 f + \lambda_2 g + \lambda_3 h$$

Tr(D^2)=Tr(A^2) gives us

$$Vk = k^2 + \lambda_1^2 f + \lambda_2^2 g + \lambda_3^2 h$$

From these equations, we use a system of equations to find f,g,h in terms of our eigenvalues.

3 Classifications

In order to further understand the distance-regular graphs they were classified, by their properties as graphs, their algebraic properties, and their multiplicities. The following terms appear in [1] and [2]

3.1 Classifications of the graphs

Distance-regular Cayley graphs of diameter 3 can be classified according to their the properties of their graph, such as bipartite, antipodal and Taylor.

Definition 7 (Bipartite graph) *A graph is bipartite if its vertices can be partitioned into two sets in such a way that no two vertices in the same set are adjacent.*

Definition 8 (Antipodal graph) *A graph of diameter* d *is said to be antipodal if, for any vertices* u, v, w *such that* $\partial(u, v) = \partial(u, w) = d$, *it follows that* $\partial(v, w) = d$ *or* $v = w$.

Definition 9 (Taylor graph) *A Taylor graph is a distance-regular graph of diameter 3 with the intersection array* $\{ k, \mu, 1 ; 1 , \mu, k\}$ *for some* $\mu \in \mathbb{Z}$.

Lemma 1 *For all Taylor graphs,* $|\Gamma_0| = 1$, $|\Gamma_1| = k$, $|\Gamma_2| = k$, *and* $|\Gamma_3| = 1$.

Proof: We know $|\Gamma_0|$ always equals 1.

Since $b_0 = k$, we know $|\Gamma_1| = k$.

Since b_1 and $c_2 = \mu$ we know $k\mu = |\Gamma_2|\,\mu$ so $|\Gamma_2| = k$

Finally since $b_2 = 1$ and $c_3 = k$, $k1 = |\Gamma_3|\,k$, so $|\Gamma_3|= 1$.
From this we also know that all Taylor graphs are antipodal.

Lemma 2 *A graph is both Taylor and bipartite if and only if its intersection array is* $\{k, k-1, 1 :, 1, k-1, k\}$

Proof: If the graph is bipartite, all $a_i = 0$ where $0 < i \leq d$. Also $a_i + b_i + c_i = k$. So $b_1 + c_1 = k$. Since $c_1 = 1$, $b_1 = k - 1$. Similarly we can find that $c_2 = k - 1$. Since it is Taylor we know $b_0 = k$, $b_3 = 1$, $c_1 = 1$, and $c_3 = k$.
If the intersection array is $\{k, k-1, 1 :, 1, k-1, k\}$ then $a_1 = a_2 = 0$ so the graph is bipartite. Additionally, the graph is Taylor since we can set $k - 1 = \mu$.

3.2 Algebraic Classifications

Many of the distance-regular Cayley graphs we found can be generated by both abelian and non-abelian groups, most commonly the cyclic and dihedral groups. In a few instances, however, we found graphs that were genuinely non-abelian.

Definition 10 (Genuinely Non-Abelian) *A Cayley graph is called genuinely non-abelian if it can only be generated by non-abelian groups.*

Many of the genuinely non-abelian groups that we found are only generated by a single group, the dihedral group. However, there are a few instances in which a distance-regular Cayley graph is generated by more than one group, all of which are non-abelian. An example of this is the graph with 30 vertices and degree 7. This is generated both by the dihedral group and also the direct product of the dihedral group of order 6 and $\mathbb{Z}_5$

In order to further classify graphs and study them, we broke them into three classifications according to their multiplicities. As mentioned in section

3.3 Multiplicities

Type 1 None of the non-trivial multiplicities are the same. $f \neq g \neq h$

Type 2 Two of non-trivial the multiplicities are the same. $f = g \neq h$

> **Note:** All bipartite graphs are Type 2 graphs (Lemma 7).

Type 3 All three of the non-trivial multiplicities are different. $f \neq g \neq h$

GAP programing is a very powerful tool in this research [4]. With some programming, it was able to develop a list of possible parameter sets and also potential multiplicities and whether the graph was Type I, II or III. The list was further refined as GAP helped determine which of the probable parameter sets for distance regular graphs of diameter 3 were Cayley graphs. Due to some limitations of GAP which exist in all computer programs, the research was limited to graphs with $v \leq 50$.

4 Findings and Theorems

This chapter will explore some distance-regular Cayley graphs that have interesting properties.

4.1 The graph generated by $\mathbb{Z}_{2p}$

Theorem 1 *When S = the set of units (U), the group $\mathbb{Z}_{2p}$, where p is prime, always forms a distance-regular graph of diameter 3 which is Taylor, bipartite and antipodal.*

We know that $U = \Gamma_1$ and $|U| = p - 1 = k$. In addition, all $u_i + u_j \notin U$ and $u_i + u_j = 0$ or $2n$ for some $n \in \mathbb{Z}$.

Lemma 3 Γ_2 *is the set of even integers 2n where ($0 < n < p$) if $p > 3$.*

Proof: Let $x = 2n$.

If $2n$-1 $\neq p$ then $x = 1 + (2n$-1).

If $2n$-1 $= p$ then $x = 3 + (2n$-3).

Therefore $x \in \Gamma_2$.
Let $y \in \Gamma_2$.

$y = u_i + u_j$.

$u_i = 2t - 1$ and $u_j = 2w$ - 1 where w, $t \in \mathbb{Z}$.

$y = 2(t$+w-1).

so $y = 2m$ where m= t + w - 1.

so $y \in 2n$.

Therefore $\Gamma_2 = 2n$ for $0 < n < p$.

Lemma 4 $\Gamma_3 = p$

Proof: p= ($2n$+1) where $2n \in \Gamma_2$ and $1 \in U$ when $n = \frac{p-1}{2}$.

So $p \in \Gamma_3$.
Let $x \in \Gamma_3$ and $x \neq p$.

$x = 2n + u_j$ for some $j \in \mathbb{Z}$.

Thus $x \in U$ except when $2n + u_k = p$.

Thus, $x \notin \Gamma_3$ unless $x = p$.

So $\Gamma_3 = p$.

Lemma 5 *Each $x \in \Gamma_2$ is adjacent to an equal number of units in Γ_1.*

Let $x \in \Gamma_2$.

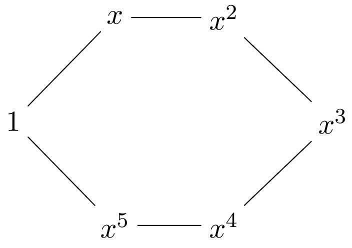
x can be written in terms of any unit, u_j in the form of x $= u_j + (x$-$u_j)$ except when $u_j + p = x$. Thus x is adjacent to every unit except u_i where $u_i + p = x$. Due to the cancellation law, $\forall\ x\ \exists!\ u_i$.
Thus $\forall\ x \in \Gamma_2$ is adjacent to exactly $k-1$ units.

Lemma 6 *The graph is antipodal, Taylor, and bipartite.*

The graph is antipodal since $|\Gamma_3| = 1$.
To be Taylor and bipartite, the intersection array must be
$\{k,\ k-1,\ 1;\ 1,\ k-1,\ k\}$ (Lemma 2)
Clearly $b_0 = k$ and $c_1 = 1$.
$a_1 = 0$ since $u_i + u_j \notin U$ so b_1 must equal $k-1$.
Both Γ_1 and Γ_2 have k elements and b_1 is $k-1$.
Therefore $k \times$ (k-1) $= k \times c_2$.
By the cancellation laws we have $k - 1 = c_2$.
$a_2 = 0$ since $2n$+ $u_i \notin 2n$, thus b_2 must equal 1.
Finally, $a_3 = 0$ since p cannot be adjacent to itself, so c_3 must equal k.
Therefore, the graph generated by the units of Z_{2p} is a Taylor, bipartite, and antipodal distance-regular graph of diameter 3.

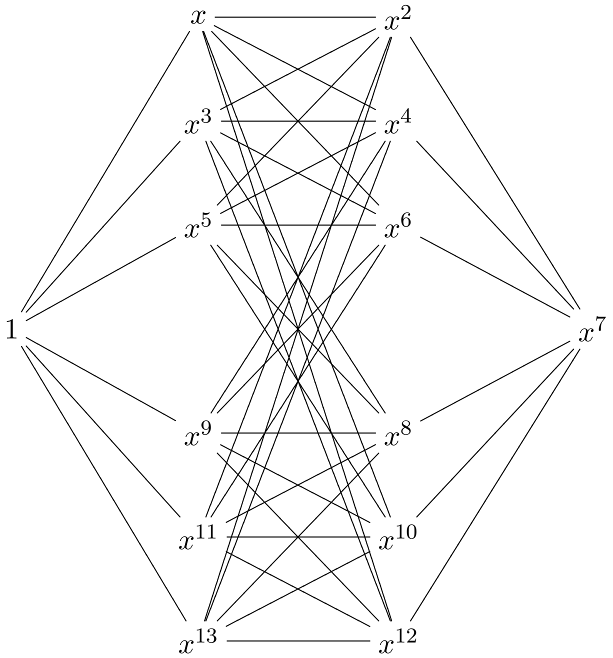
Below are some examples of graphs that belong to this family of $\mathbb{Z}_{2p}$.

Group $\mathbb{Z}_{2*3}$ and degree 2
S= $\{x,\ x^5\}$



Eigenvalues: -1, 1, 2, -2
Multiplicities: 2, 2, 1, 1

Group $\mathbb{Z}_{2*7}$ and degree 6
S=$\{x$,x^3,x^5,x^9,x^{11},$x^{13}\}$



Eigenvalues: -1, 1, -6, 6
Multiplicities: 6, 6, 1, 1

4.2 Taylor and Bipartite graphs

The family of graphs formed by $\mathbb{Z}_{2p}$ is also interesting since all the multiplicities of the graph correlate to the sizes of the associate classes. This property holds for all graphs that are Taylor and bipartite.

Theorem 2 *If a distance regular diameter 3 graph is both Taylor and bipartite then the multiplicities of its eigenvalues will reflect the sizes of the associate classes and will be 1, k, k, and 1.*

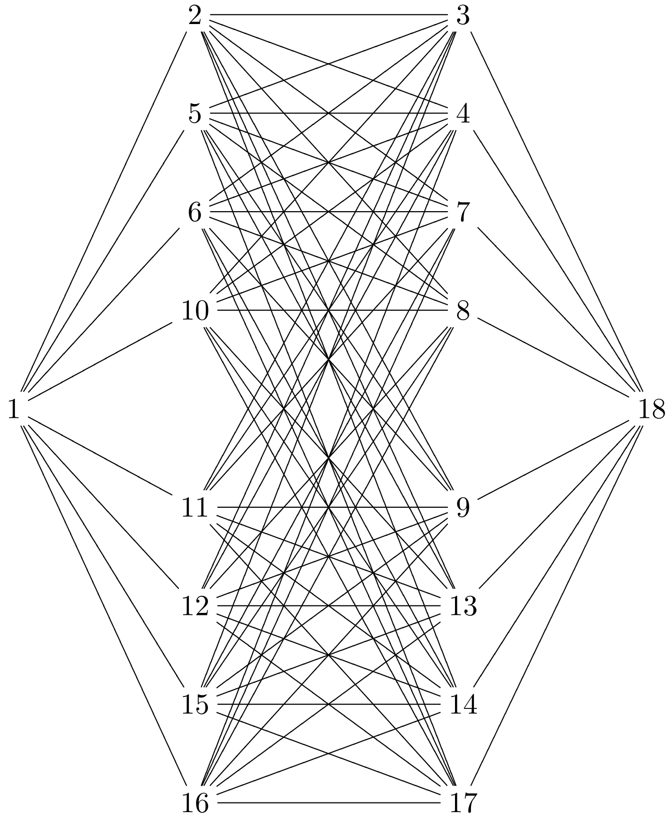
Lemma 7 *-k is an eigenvalue if and only if the graph is bipartite. If so, then for every eigenvalue θ, -θ is also an eigenvalue with the same multiplicity.*

Proof: This is shown in Proposition 3.2.3 of [2]

From section 2.2 we know k is an eigenvalue with multiplicity 1. Lemma 7 shows that $-k$ must also be an eigenvalue with multiplicity 1. These two multiplicities correlate to Γ_0 and Γ_3 since in a Taylor graph, $|\Gamma_0|$ and $|\Gamma_3| = 1$. (Lemma 1) Furthermore, Lemma 7 shows that $\exists$ eigenvalues θ and -θ such that both have the same multiplicity, f, where $f \in \mathbb{Z}$. The sum of the multiplicities $= v$, so $2f = v - 2$.

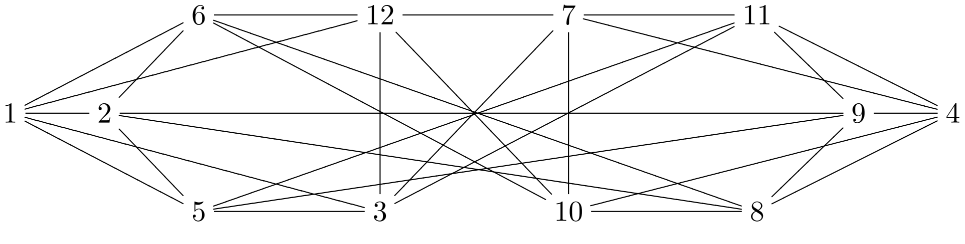
> Moreover, $|\Gamma_1|$ and $|\Gamma_2| = k$ (Lemma 1) so it follows that $2k = v - 2$. Thus $2k$=$2f$ and by the cancellation laws, k=f. Therefore, if a graph is both Taylor and bipartite, its multiplicities will always be 1, k, k, and 1 which reflect the sizes of the associate classes.

Since all graphs from the family of $\mathbb{Z}_{2p}$ are Taylor and bipartite, the graphs in the last section demonstrate this property. However, there are groups that are not a member of $\mathbb{Z}_{2p}$ and still Taylor and bipartite. Below is a graph of the Dihedral group of order 18 and degree 8 which is both Taylor and bipartite.



Eigenvalues: -1 , 1 , -8 , 8
Multiplicities: 8 , 8 , 1 , 1

The graphs must be both Taylor and bipartite as shown by the following example in which the graph is Taylor but not bipartite. It is the alternating group of order 12 and degree 5.



Eigenvalue: $\sqrt{5}$,$-\sqrt{5}$,-1, 5
Multiplicity: 3 , 3 , 5 , 1

In the following section, we will see graphs that are bipartite but not Taylor and thus all the multiplicities will not reflect all the associate classes.

4.3 Dihedral groups

A correlation between the multiplicities and the sizes of the associate classes is not unique only to Taylor and bipartite graphs, but also occur to a smaller extent in dihedral groups.

Theorem 3 $|\Gamma_2|$ *always equals the multiplicity* $\neq 1$ *for all distance-regular Cayley graphs on dihedral groups.*

Lemma 8 *Every non-trivial distance-regular Cayley graph on a dihedral group is bipartite and diameter 3.*

Proof: This is stated in [6] (Theorem 1.3)

Lemma 9 $|\Gamma_0|$ + $|\Gamma_2|$ $=$ $|\Gamma_1|$ + $|\Gamma_3|$

We know $|\Gamma_0|= 1$, $|\Gamma_1| = k$, $b_0 = k$, c_1=1, a_1=0, a_2=0 and a_3=0. From this we can find that $b_1 = k$-1 and $c_3 = k$.
With this information we can derive:

$$k(k-1) = c_2\,|\Gamma_2| \tag{4}$$

$$k\,|\Gamma_3| = b_2\,|\Gamma_2| \tag{5}$$

$$c_2 = k - b_2 \tag{6}$$

from 4 5 and 6 we get

$$k(k-1) = (k-b_2)\frac{k\,|\Gamma_3|}{b_2} \tag{7}$$

from 7 and 6 we get

$$|\Gamma_3| = \frac{(k-c_2)(k-1)}{(c_2)} \tag{8}$$

if we add $|\Gamma_1|$ or k to 8 we get

$$|\Gamma_3| + k = \frac{k^2-k+c_2}{c_2} \tag{9}$$

if we solve for $|\Gamma_2|$ in 5 and then add $|\Gamma_0|$ or 1 we get

$$|\Gamma_2| + 1 = \frac{k^2 - k + c_2}{c_2} \tag{10}$$

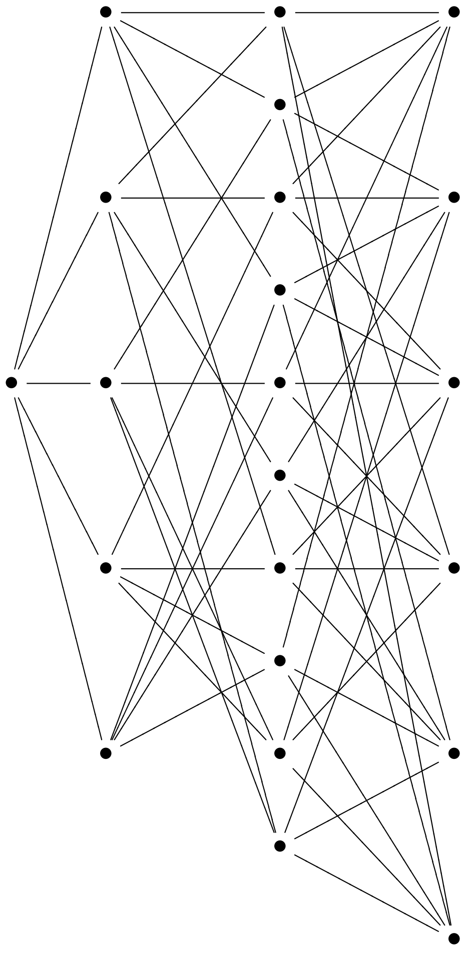
Thus from 9 and 10 we have

$$|\Gamma_0| + |\Gamma_2| = |\Gamma_1| + |\Gamma_3| \tag{11}$$

Thus we know that $|\Gamma_0| + |\Gamma_2| = |\Gamma_1| + |\Gamma_3| = \frac{v}{2}$
$|\Gamma_0| = 1$ so $|\Gamma_2| = \frac{v}{2}$-1

From Lemma 7 and Lemma 8 know that $\exists$ double eigenvalues and double multiplicities (1 and f where f $\in \mathbb{Z}$)
Therefore 2f=v-2 or f= $\frac{v}{2}$ -1 = $|\Gamma_2|$.
Thus f=$|\Gamma_2|$

One example of this theorem is the Dihedral Group of order 12 and degree 5.



5 An Interesting Graph

Currently, only three strongly regular graphs are known that have μ =1. These three graphs are the pentagon, the Peterson graph, and the Hoffman-Singleton graph. Previously, the smallest open case was the (400,21,2,1) strongly regular graph. This report explains some necessary characteristics of the graph.

5.1 Definitions

Definition 11 (Strongly Regular Graph) *A (V, k, λ, μ) strongly regular graph is a distance-regular graph with V vertices, adjacent to exactly k other vertices. It must also have the property that if two vertices are adjacent, they share exactly λ vertices. If the two vertices are not adjacent they must share exactly μ vertices.*

5.2 Proof

Start with any vertex. We know this vertex is adjacent to exactly 21 points. So $|\Gamma_1|$ is 21. Since $|\Gamma_0|+|\Gamma_1|+|\Gamma_2|= 400$, $|\Gamma_2|$ must be 378.
Certain values from the intersection array are known.

$$\begin{array}{cc} b_0=k=21 & c_1=1 \\ |\Gamma_0|=1 & |\Gamma_1|=21 \end{array}$$

Using this information, the values of a_1, a_2, b_1 and c_2 can be found. Because of the properties of the intersection array, we know that $\Gamma_2 = \frac{\Gamma_1 * b_2}{c_2}$. So $378 = \frac{21 * b_2}{c_2}$. If $c_2 > 1$, then $b_2 \geq 36$, which is impossible, so the only possible solutions are b_1=18 and c_2=1. Since $a_i + b_i + c_i$=k, a_1=2 and a_2=20.

In Γ_1, the 21 vertices must be adjacent to exactly 2 others in Γ_1.

Lemma 10 *If two vertices in Γ_1, x and y, are adjacent, they must each be adjacent to exactly one more vertex in Γ_1. In fact, they must be adjacent to the same vertex.*

Proof: Let x and y be two adjacent vertices in Γ_1. Assume x is adjacent to z and y is adjacent to w. Assume $z \neq w$. Since μ=1 and x and w are not adjacent, they must share exactly 1 vertex. x and w share both Γ_0 and x, which is a contradiction.

Therefore, Γ_1 is composed of 7 disjoint triangles.

In Γ_1 we label the 21 vertices A_0, B_0,..., U_0. Because c_2=1 and b_1=18, each vertex in Γ_1 is adjacent to 18 unique vertices in Γ_2.

Definition 12 (W-class) *A W-class consists of all of the points in* Γ_2 *that are adjacent to same vertex* W_0 *in* Γ_1*, where W is from A...U. Each W-class consists of 18 points* $W_1 ... W_{18}$*. Two classes, X and Y, are considered adjacent if the vertices* X_0 *and* Y_0 *in* Γ_1 *are adjacent.*

Lemma 11 *If two vertices in* Γ_2*,* X_1 *and* X_2*,* X_1 *is adjacent to exactly one more vertex in the same W-class.*

Proof: Assume X_1 is adjacent to three vertices X_2,X_3,X_4 in the same W-class. X_1 and X_0 are adjacent, so by λ=2, we know they must share exactly 2 vertices. They share X_2,X_3,X_4, which is a contradiction, so X_1 is adjacent to at most one vertex beside X_2.

Assume X_1 is adjacent to only X_2 in the X-class. X_1 and X_2 are adjacent,so they must share 2 vertices. X_1 and X_2 are both adjacent to X_0. If X_1 is adjacent to only $X - 2$ in the X-class, then both X_1 and $X - 2$ so must both be adjacent to 1 more vertex in a non-adjacent W-class (See Lemma 13). This is a contradiction that is proven later in Lemma 14. So X_1 must be adjacent to at least one more vertex beside X_2 in the X-class.

Therefore, if X_1 and X_2 are adjacent X_1 must be adjacent to exactly 1 more vertex in the same W-class.

Lemma 12 *If* X_1 *is adjacent to* X_3 *and* X_2 *is adjacent to* X_4*,* X_3*=*X_4*.*

Proof: Let X_1 and X_2 be adjacent vertices in the same W-class. Let X_1 be adjacent to X_3 and X_2 be adjacent to X_4. Assume $X_3 \neq X_4$. Then X_1 is not adjacent to X_4, so by μ=1, X_1 and X_4 must share exactly one vertex. They share X_0 and X_2, which is a contradiction. So, X_3=X_4.

Therefore, each W-class in Γ_2 is composed of 6 disjoint triangles.

Lemma 13 *If two points in* Γ_2 *are in adjacent W-classes, they cannot be adjacent to each other.*

Proof:

Let X_1 and Y_1 be points in two adjacent W-classes, $W(X)$ and $W(Y)$ and let X_1 be adjacent to Y_1. X_0 will be non-adjacent to Y_1, so by μ=1 they must share exactly one vertex. Both are adjacent to X_0 and Y_0, which is a contradiction.

Lemma 14 *A vertex* X_n *,* $1 \leq n \leq 18$ *in the X-class can be adjacent to only one vertex in each non-adjacent W-class.*

Proof: Assume a vertex X_1 is adjacent to two points, Y_1 and Y_2, in a non-adjacent W-class. If Y_1 and Y_2 are adjacent, then Y_1 and Y_2 share a third vertex in the Y-class(by Lemmas 10),X_1, and Y_0 which is a contradiction of λ=2. If Y_1 and Y_2 are not adjacent, they share the vertices X_1 and Y_0, which is a contradiction of μ=1.

Therefore, a vertex in one W-class can only be adjacent to 1 vertex in another W-class.

Lemma 15 *If two vertices X_i and Y_i are adjacent, then no vertex adjacent to X_i in the X-class can be adjacent to any vertex adjacent to Y_i in the Y-class.*

Proof: Assume X_i is adjacent to X_j, X_h and Y_i. Y_i is adjacent to Y_j and Y_h. Assume X_j is adjacent to Y_j. By Lemma 14, we know that X_i cannot be adjacent to Y_j, so they must share exactly 1 vertex. However, they will share both X_j and Y_i, which is contradictory.

By using this, it should be possible to rule out the existence of the (400,21,2,1) graph. If it is ruled out in this way, it will be independent of the value of k, and so all graphs with λ=2 and μ=1 would be ruled out.

6 Appendix

In the appendix, there is a list of possible parameter set for distance-regular Cayley graphs. In the excel v is the order of the group, or number of vertices in the graph. The cn is the catalog number which GAP used to recognize that group (always denoted as 'SmallGroup(v, cn)' in GAP). The name of the group if known at the time to the authors. Whether the graph is Cayley and the degree (k) of the graph. Some of the parameter do not have a cn number listed, but rather 'prob with GAP'. This means that GAP either ran out of memory, could not extend its workspace any further, or did not have enough time to finish. (The authors ran computers for 5 straight days, but even after that GAP had not finished looking in some groups. All the graphs are of diameter 3, and all are believed to be Type 11 with the exception of v= 27 and cn= 4 and k=6.

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