

# WALK-REGULAR GRAPHS THAT ARE NOT VERTEX-TRANSITIVE

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ABSTRACT. We explore relationships between the geometric structure of finite graphs and the linear algebra structure of their associated adjacency matrix. We do this by examining regular graphs which are walk-regular but not vertex-transitive, focusing on graphs with a small number of distinct eigenvalues. We examine the eigenvectors, eigenvalues and cycle structure of graphs with four or five eigenvalues. Seidel switching is used to create new graphs. This is joint work with Alan LaMielle, under the supervision of Ken W. Smith.

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## 1. BACKGROUND

**Definition 1.1.** A **graph**  $G = (V, E)$  consists of a **vertex set**,  $V(G) = \{v_1, \dots, v_n\}$ , and an **edge set**,  $E(G) = \{e_1, \dots, e_j\}$ . Each edge,  $e_i \in E(G)$ , is a pair of vertices,  $e_i = \{(v_a, v_b) : v_a, v_b \in V(G)\}$ .

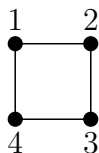
**Definition 1.2.** Let  $G$  be an undirected, unweighted graph.  $G$  is a **simple graph** if each  $e_i \in E(G)$  is unique and no edge is of the form  $(v_i, v_i)$ . That is, no multiple edges or loops are allowed in the graph.

**Definition 1.3.** The **adjacency matrix** of a graph is the square,  $n \times n$  matrix with a one in the  $(i, j)$ -position if  $v_i$  is adjacent to  $v_j$ , and zeroes everywhere else.

**Definition 1.4.** The **spectrum** of a graph  $G$  is the set of eigenvalues,  $\lambda$ , of the adjacency matrix of  $G$ , along with their multiplicities,  $m$ :

$$\text{Spec}(G) = \{(\lambda_1)^{m_1}, (\lambda_2)^{m_2}, \dots, (\lambda_i)^{m_i}\}$$

An example of a simple graph, its adjacency matrix, and its spectrum are as follows ...



$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{Spec}(G) = \{(2)^1, (0)^2, (-2)^1\}$$

**Definition 1.5. Algebraic Graph Theory** uses the algebraic properties of a graph (for instance, its adjacency matrix) to discover new information about the graph.

**Definition 1.6.** A graph that is **walk-regular** has the property that every vertex is on exactly the same number of 3,4,...,n-cycles.

**Definition 1.7. Vertex-Transitive:** Given a graph  $G$  with vertex set  $V(G)$ ,  $\forall x, y \in V(G)$ ,  $\exists \sigma \in \text{Aut}(G)$  such that  $\sigma(x)=y$ .

The graph in the above example is both walk-regular and vertex-transitive.

## 2. WALK-REGULAR GRAPHS

**Question.** What graphs are walk-regular?

**Theorem 2.1.** All regular graphs with four distinct eigenvalues are walk-regular [4].

*Proof.* The minimal polynomial of a graph  $G$  having adjacency matrix  $A$  and four distinct eigenvalues,  $\lambda_1 = k, \lambda_2, \lambda_3, \lambda_4$ , is defined as,

$$m(x) = (x - k)(x - \lambda_2)(x - \lambda_3)(x - \lambda_4)$$

and has the property that

$$m(A) = (A - kI)(A - \lambda_2I)(A - \lambda_3I)(A - \lambda_4I) = 0$$

and

$$m(A) = (A - kI)(A^3 - (\lambda_2 + \lambda_3 + \lambda_4)A^2 + (\lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4)A - (\lambda_2\lambda_3\lambda_4)I) = 0$$

Let

$$\begin{aligned}\alpha_0 &= (\lambda_2 + \lambda_3 + \lambda_4) \\ \alpha_1 &= (\lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4) \\ \alpha_2 &= (\lambda_2\lambda_3\lambda_4)\end{aligned}$$

Then

$$m(A) = (A - kI)(A^3 - \alpha_0A^2 + \alpha_1A - \alpha_2I) = 0$$

The goal here is to find an equation for  $A^3$  in terms of smaller powers of  $A$ .

Let  $\vec{j}$  be the all ones vector with eigenvalue  $k$  such that  $A\vec{j} = k\vec{j}$ . Then

$$(A - kI)\vec{j} = 0$$

but

$$(A^3 - \alpha_0A^2 + \alpha_1A - \alpha_2I) \neq 0$$

Let  $\vec{v}$  be an eigenvector with eigenvalue  $\lambda \neq k$  such that  $A\vec{v} = \lambda\vec{v}$ . Then

$$(A^3 - \alpha_0A^2 + \alpha_1A - \alpha_2I)\vec{v} = 0$$

Let

$$\beta = \frac{(k - \lambda_2)(k - \lambda_3)(k - \lambda_4)}{n}$$

Then

$$[(A^3 - \alpha_0A^2 + \alpha_1A - \alpha_2I) - \beta J]\vec{j} = 0$$

since  $\beta J\vec{j} = \beta n\vec{j}$ .

Also,

$$[(A^3 - \alpha_0A^2 + \alpha_1A - \alpha_2I) - \beta J]\vec{v} = 0$$

since  $\vec{v} \perp \vec{j} \implies \beta J\vec{v} = 0$ .

We now have a cubic matrix equation that sends every vector to zero. So,

$$\begin{aligned}A^3 - \alpha_0A^2 + \alpha_1A - \alpha_2I - \beta J &= 0 \\ \implies A^3 &= \alpha_0A^2 - \alpha_1A + \alpha_2I + \beta J\end{aligned}$$

Since the diagonal entries of  $A^2, A, I, J$  are constant, we need only to find

$$\alpha_0k + \alpha_2 + \beta$$

to know how many walks of length three are from  $v_i$  to  $v_i$ . This number will be twice the number of triangles on each vertex. Multiplying both sides of the cubic equation by  $A$  will give a recursive formula for the number of walks of length 4, 5, . . . etc. on each vertex. Modifying this

answer by subtracting the number of trivial walks and dividing by the number of times the same cycle was counted will give the desired answer.  $\square$

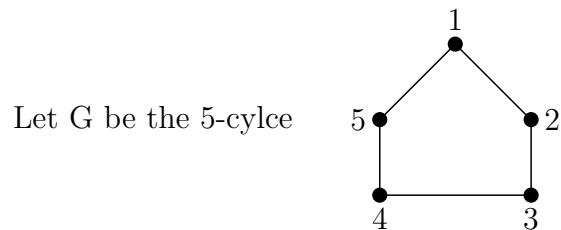
In order to determine if a graph with five distinct eigenvalues is walk-regular, we would need to know if it had constant diagonal entries in  $A^3$ . For a graph with six distinct eigenvalues, the diagonal entries of  $A^3$  and  $A^4$  are needed, etc.

### 3. SEIDEL SWITCHING

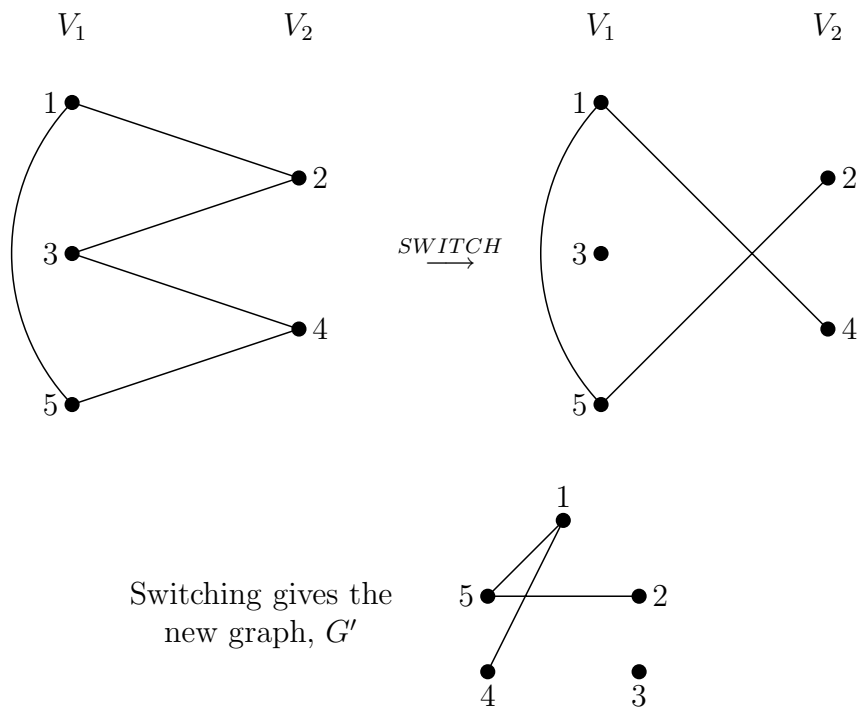
We were looking for a way to systematically alter known graphs [4] with four distinct eigenvalues in order to examine exactly how their properties changed.

**Definition 3.1.** *Seidel Switching* partitions the vertices into two sets,  $V_1$  and  $V_2$  and switches adjacencies with respect to this partition. Adjacency is maintained within the partition sets, and complemented between the partition sets.

The following example demonstrates the basic principle behind Seidel switching.



Switching with respect to partition sets,  $V_1 = \{1, 3, 5\}$  and  $V_2 = \{2, 4\}$  looks like ...



However,  $G'$  is not a regular graph. Since we are only concerned with regular graphs, we need only to look at *regular* partitions.

**Definition 3.2.** Two partition sets are said to be **regular** if and only if the induced subgraph with vertex set  $V_i$  is regular, and every vertex in one partition is adjacent to a constant number of vertices in the other partition.

Even though it is likely that  $\text{Spec}(G) \neq \text{Spec}(G')$ , the graphs will be cospectral with respect to their *Seidel matrices*.

**Definition 3.3.** The **Seidel Matrix** is defined as  $S = J - 2A - I$ . This gives a 0 on the diagonal, a  $-1$  when  $v_i \sim v_j$ , and a 1 when  $v_i \not\sim v_j$ .

**Theorem 3.1.** A graph  $G$  and a graph  $G'$  found by Seidel switching on  $G$  are cospectral with respect to their Seidel matrices.

*Proof.* Eigenvalues of  $S$  are found by its characteristic polynomial,  $\det(S - \lambda I)$ . Let  $U$  be an  $n \times n$  matrix with diagonal entries  $(v_i, v_i) = -1$  if  $v_i \in V_1$ , and  $(v_i, v_i) = 1$  if  $v_i \in V_2$ . Multiplying  $S$  on the left and the

right by  $U$  and  $U^{-1}$  (where  $U = U^{-1}$ ) is equivalent to complementing adjacencies between vertex partitions. So  $USU^{-1} = S'$ , where  $S'$  is the Seidel matrix of the new graph  $G'$ . To find the eigenvalues of  $S'$  we need to look at its characteristic polynomial,

$$\begin{aligned} & \det(S' - \lambda I) \\ &= \det(USU^{-1} - \lambda I) \\ &= \det(U)\det(S - \lambda I)\det(U^{-1}) \\ &= \det(S - \lambda I) \end{aligned}$$

$\therefore S$  and  $S'$  are cospectral. □

This result leads us to the following lemma.

**Lemma 1.** *If  $\vec{v}$  is an eigenvector of  $S$  with eigenvalue  $\lambda$ , then  $U\vec{v}$  is an eigenvector of  $S' = USU^{-1}$  with eigenvalue  $\lambda$ .*

*Proof.* Suppose  $S\vec{v} = \lambda\vec{v}$ .

$$\begin{aligned} S'(U\vec{v}) &= USU^{-1}(U\vec{v}) \\ &= US\vec{v} \\ &= U(\lambda\vec{v}) \\ &= \lambda(U\vec{v}) \end{aligned}$$

□

We also know the exact relation between  $Spec(G)$  and  $Spec(S)$ .

**Corollary 1.** *Switching with respect to a regular partition changes at most two of the eigenvalues of the graph,  $G$  [4].*

*Proof.* Given the Seidel matrix,  $S = J - 2A - I$ , suppose  $\vec{v}$  is an eigenvector of  $A$  such that  $A\vec{v} = \lambda\vec{v}$ . Then,

$$\begin{aligned} S\vec{v} &= (J - 2A - I)\vec{v} \\ &= J\vec{v} - 2A\vec{v} - I\vec{v} \\ &= J\vec{v} - 2\lambda\vec{v} - \vec{v} \\ &= J\vec{v} - (2\lambda - 1)\vec{v} \end{aligned}$$

Case 1:  $\vec{v} = \vec{j}$ , the all ones vector such that  $A\vec{j} = k\vec{j}$  and  $J\vec{j} = n\vec{j}$ .  
Then

$$\begin{aligned}
S\vec{j} &= n\vec{j} - (2k+1)\vec{j} \\
&= (n-2k-1)\vec{j} \\
\therefore S &= n-2k-1
\end{aligned}$$

Case2:  $\vec{v} = \langle \vec{j} \rangle^\perp$ , any other eigenvector orthogonal to  $\vec{j}$  such that  $A\vec{v} = \lambda\vec{v}$  and  $J\vec{v} = \vec{0}$ . Then

$$\begin{aligned}
S\vec{v} &= \vec{0} - (2\lambda+1)\vec{v} \\
\therefore S &= -(2\lambda+1)
\end{aligned}$$

So when  $G$  is a regular, connected graph on  $n$  vertices with eigenvalues,  $\lambda_1 = k > \lambda_2, \dots, > \lambda_n$ , the eigenvalues of  $S$  are,  $\alpha_1 = n - 2k - 1, \alpha_2 = -2\lambda_2 - 1, \dots, \alpha_n = -2\lambda_n - 1$ .

This relationship is also true for  $G'$ .

If  $Spec(G') = \{\omega_1 = (K)^1, (\omega_2)^{M_2}, \dots, (\omega_n)^{M_n}\}$ , then  $Spec(S') = \{(n - 2K - 1)^1, (-2\omega_2 - 1)^{M_2}, \dots, (-2\omega_n - 1)^{M_n}\}$ .

This presents two cases for possible spectra of  $G'$ .

**Case 1:**  $n - 2K - 1 = n - 2k - 1$  and  $-2\omega_i - 1 = -2\lambda_i - 1$ ,  
 $\forall \omega \in Spec(G')$  and  $\forall \lambda \in Spec(G)$ .

This gives a graph that is cospectral and possibly nonisomorphic to  $G$ .

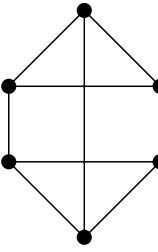
**Case 2:**  $n - 2K - 1 = -2\lambda_i - 1$ ,  $-2\omega_i - 1 = n - 2k - 1$ , and  
 $-2\omega_j - 1 = -2\lambda_j - 1$ .

**a:** If  $m_i = 1$  for  $-2\lambda_i - 1$ , then  $G'$  will have either three or four distinct eigenvalues. If  $G'$  has only three distinct eigenvalues, then  $n - 2k - 1 = -2\lambda_g - 1$  for some  $\lambda_g \in Spec(G)$ .

**b:** If  $m_i > 1$  for  $-2\lambda_i - 1$ , then  $G'$  will have either four or five distinct eigenvalues. If  $G'$  has five distinct eigenvalues, then  $n - 2k - 1 \neq -2\lambda_g \forall \lambda_g \in Spec(G)$ .

In both **a** and **b** of Case 2, there is only possibility for at most two eigenvalues of  $G$  to change.  $\square$

To illustrate this proof, take for example the following graph,

$$\begin{aligned}
 \text{Spec}(G) = & \\
 \{(3)^1, (1)^1, (0)^2, (-2)^2\} &
 \end{aligned}$$


This gives  $\text{Spec}(S) = \{(-1)^1, (-3)^1, (-1)^2, (3)^2\} = \text{Spec}(S')$ .  
 Since  $n - 2k - 1 = -1 = -2\lambda_3 - 1$ ,  $\text{Spec}(G')$  can have only three or four distinct eigenvalues.

**Case 1:**  $n - 2K - 1 = -1 \implies \text{Spec}(G') = \text{Spec}(G).$

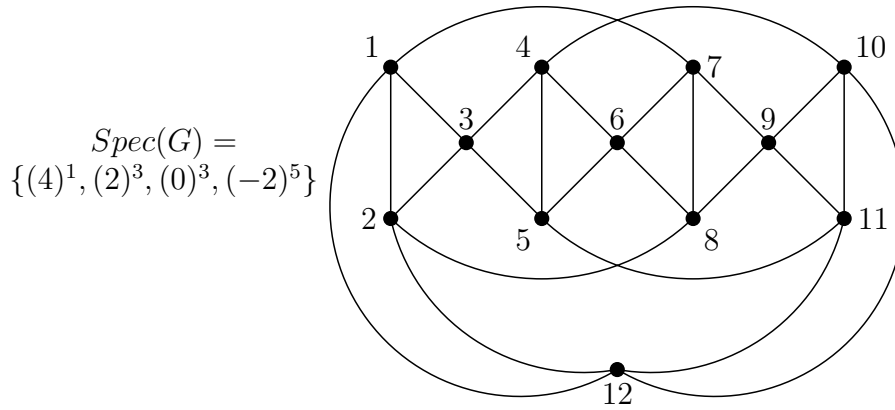
**Case 2:**  $n - 2K - 1 = -3 \implies \text{Spec}(G') = \{(4)^1, (0)^3, (-2)^2\}$

**Case 3:**  $n - 2K - 1 = 3 \implies \text{Spec}(G') = \{(1)^2, (0)^3, (-2)^1\}$

Given what we know about regular partitions and the relationship between  $\text{Spec}(G)$  and  $\text{Spec}(S)$ , we are able to make conclusions about possible switching sets for  $G$ . This gives us a systematic way to manipulate and examine the graphs we are interested in.

#### 4. MAIN RESULTS

Since we are interested in walk-regular, non vertex-transitive graphs, we examined possible switching sets on the smallest known graph of this type [2] in order to determine if it would give us a new walk-regular, non vertex-transitive graph.



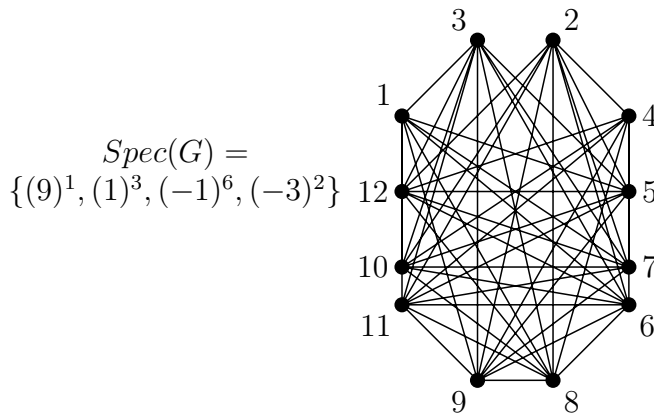
With the relation  $Spec(G) \longrightarrow Spec(S) \longleftarrow Spec(S') \longrightarrow Spec(G')$ , the above graph gives these possibilities for  $G'$  ...

- (1)  $Spec(G') = \{(4)^1, (2)^3, (0)^3, (-2)^5\}$
- (2)  $Spec(G') = \{(6)^1, (2)^3, (0)^2, (-2)^6\}$
- (3)  $Spec(G') = \{(8)^1, (2)^2, (0)^3, (-2)^6\}$

None of the above spectra permit a new  $G'$ .

**Question.** *Is it possible that switching on other walk-regular, non vertex-transitive graphs will give a new  $G'$ ?*

Of the other 13 graphs on 6-12 vertices with four integral eigenvalues [4] (which are all vertex-transitive) that Seidel switching was done on, only one gave us new information.



From the spectra restrictions on  $G, S, S'$ , and  $G'$ , we found two feasible spectra for  $G'$ ,

$$(1) \operatorname{Spec}(G'_1) = \{(7)^1, (3)^1, (1)^2, (-1)^6, (-3)^2\}$$

$$(2) \operatorname{Spec}(G'_2) = \{(5)^1, (3)^1, (1)^3, (-1)^5, (-3)^2\}$$

(1) comes from switching on the vertices of  $G$  with respect to the partitions  $V_1 = \{1, 3, 6, 8, 10, 12\}$ , and  $V_2 = \{2, 4, 5, 7, 9, 11\}$ .

(2) comes from switching on the vertices of  $G$  with respect to the partitions  $V_1 = \{1, 2, 5, 6, 9, 10\}$ , and  $V_2 = \{3, 4, 7, 8, 11, 12\}$ .

Even though both new graphs have more than four distinct eigenvalues, they are both walk-regular, and they are both vertex-transitive.

## 5. ONGOING RESEARCH

- Under what conditions do walk-regular graphs with more than four distinct eigenvalues exist?
- Under what conditions are these graphs not vertex-transitive?
- It is likely that an infinite number of walk-regular, non vertex-transitive graphs exist. Are there certain graph constructions (Seidel switching, product constructions, etc.) that when done on a graph of this type, will always produce a graph of this type?

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