ZERO-DIVISOR SEMIGROUP GRAPHS

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Abstract. A graph is associated to a commutative zero-divisor semigroup where the vertices are the non-zero zero divisors of the semigroup. We study complement graphs associating them to commutative zero-divisor semigroups where there exists an edge between two distinct vertices x,y if and only if $x \cdot y \neq 0$. Specifically we look at graphs whose complement can be associated to a zero-divisor semigroup by following certain specified conditions.

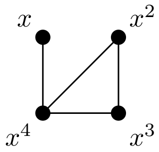
Given a compact surface M there exists a class of triangulations $C(T_i)$ where each T_i is a graph associated to a zero-divisor semigroup. We identify the elements of $C(T_i)$ and explore their properties. We also study the clique homology on T_i and compute several examples.

Contents

1. INTRODUCTION TO ZERO-DIVISOR SEMIGROUPS

A given graph G is said to represent a semigroup where the vertices of G are the non-zero elements of the semigroup and an edge is drawn between two vertices x and y if $x \cdot y = 0$. A given graph G_z is said to represent a zero-divisor semigroup where each vertex is a zero divisor, and an edge is drawn between two vertices x and y *iff* $x \cdot y = 0$.

Example 1. *Given the the semigroup* $S = \{x, x^2, x^3, x^4, x^5 = 0\}$, *associate to* S *the following zero-divisor semigroup graph:*



One can see from the above graph that there is an edge between x_i *and* x_j $\iff$ $x_i \cdot x_j$*=0.*

Definition 1. *A graph* $\overline{G}$ *is said to be the complement of a graph* G *if* $\overline{G}$ *has the same vertex set as* G *where the edges of* G *are non-edges in* $\overline{G}$ *and the non-edges of* G *are edges of* $\overline{G}$*.*

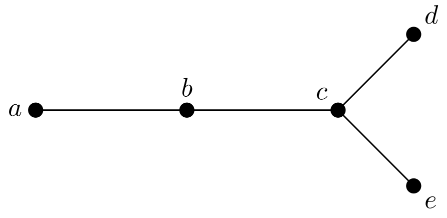
Example 2.

The above graphs are examples of a graph G *and it's complement* $\bar{G}$*.*

Definition 2. *A graph* G *is a tree iff* G *is simple, connected, and contains no cycles.*

Definition 3. *A leaf is a vertex of degree 1.*

Example 3. *In the following example vertices* a, d*, and* e *would be considered leaves.*



Definition 4. *Let* v *be a vertex in a graph* G*. The neighbors of* v *consist of the set of all vertices adjacent to* v *denoted* $N(v)$*.*

Example 4. *Looking at the graph in the previous example, one can see that the N(c)=*$\{b, d, e\}$.

Definition 5. *The diameter of a graph is defined as maximum value of the minimum distance between any two vertices in the graph, and is denoted diam(G).*

Definition 6. *The degree of a vertex* v *is the number of edges incident with* v*. i.e. In example 2, the degree of* c *is 3.*

Definition 7. *The core of a graph* G *is the induced subgraph on all vertices of* G *except those that are degree one.*

Theorem 1. [4][3]*If* G_Z *is the graph of a semigroup then* G_Z *satisfies all of the following conditions.*

1. *The graph* G_Z *is a simple connected graph. i.e. if there are two vertices* x *and* y *then there cannot be an edge from* x *to itself nor can there be more than one edge from* x *to* y.
2. *The* $diam(G_Z) \leq 3$.
3. *The graph* G_Z *consistes of a core together with vertices of degree one and the core of* G *is a union of triangles and quadrilaterals.*
4. *If* x *and* y *are any two non-adjacent vertices, then* $\exists$ *a vertex* z *with* $N(x) \bigcup N(y) \subseteq \overline{N(z)}$.

Theorem 2. *If* G *is a graph with* n *vertices and at least one vertex* x_1 *that is adjacent to* $x_2, x_3, \ldots, x_n$ *then* $\exists$ *a commutative zero-divisor semigroup that can be associated with* G.

Proof. Set all non-zero products equal to x_1. Make x_1 two-step nilpotent. Set all other squares equal to x_1. Now take two non-adjacent vertices x_i and x_j. $x_i \cdot x_j = x_1$. Multiply both sides by another vertex say x_k. Where x_k need not be distinct.

$$\begin{gathered} x_k \cdot (x_i \cdot x_j) = x_1 \cdot x_k \\ x_k \cdot (x_1) = 0 \\ 0 = 0 \end{gathered}$$

Since x_1 is adjacent to every vertex, x_1 multiplies every other variable to zero. Therefore, any graph with a vertex adjacent to every other vertex has a communtative zero-divisor semi-group associated to it.

□

2. The Complement of a Tree

Definition 8. *Let* G *be a tree, and let* L *be the set of degree 1 vertices in* G*. Define the tree depth of a vertex* $a \in V(G)$ *to be* $Tdepth(a|G) = min\{d(a, z)|z \in L\}$*. When context is clear, we also abbreviate* $Tdepth(a|G)$ *as just* $Tdepth(a)$.

Definition 9. *A vertex is called a critical vertex if it is adjacent to 2 leaves or it is adjacent to a vertex of depth 2.*

Definition 10. *A set of verticies in a connected graph is said to satisfy the even path property if each vertex is an even number of steps away from all other vertices in that set.*

Conjecture 1. *A tree is a semigroup if and only if the set of all critical vertices satisfy the even path property and no edge has depth* > 2. *The set of internal points contains all vertices that are either adjacent to a point of depth 2, or are adjacent to more than one leaf.*

2.1. Algorithm for Assigning a Semigroup to a Tree that is Admissible.

STEP 1: *First of all, assign the squares of internal vertices according to the following rules.*

1. *Depth 2 vertices are assigned to be 2-step nilpotent.*
2. *Vertices that are around depth 2 points are assigned idempotent.*
3. *Vertices of depth 1 that have more than one leaf are assigned idempotent.*
4. *Let* x *be one of the remaining internal vertices in* G*: if* x *is adjacent to an idempotent internal vertes, then* x^2 *can be any of the leaves adjacent to* x *(i.e.* x *is transpotent). If* x *is adjacent to a transpotent vertex then* x *must be idempotent.*

STEP 2: *Next define the product of adjacent vertices as follows:*

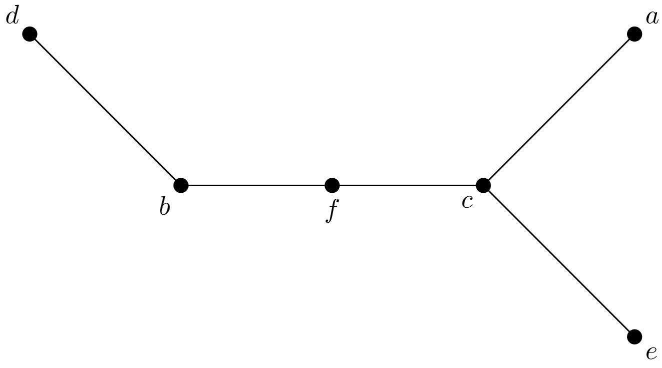
1. *The product of a vertex* a *and another* b*, where depth(a)=2, and depth(b)=1, is any leaf incident with* b.
2. $a \cdot b$*, where depth(a)=1, and depth(b)=1, and* a *is idempotent, is any leaf adj(a).*
3. $a \cdot b$*, where depth(a)=1, and* b *is a leaf adj(a), is* b.

STEP 3: *Finally, define the square of two leaves as follows:*

1. *For any two internal vertices,* a *and* b*, their product must be a leaf.*
2. *Define* $(a \cdot b)^2=0$*, for all* a *and* b.
3. *If the leaf is connected to an internal vertex that is transpotent, then that leaf is idempotent.*
4. *All the other leaves' square can be either themselves or zero.*

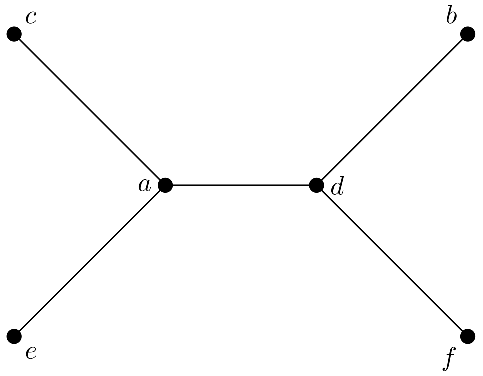
3. EXAMPLES ON SIX VERTICES

Example 5. *Consider the following graph:*



In the above example, we were able to conclude that f^2 *would be forced to be* $2 -$ *stepnilpotent. This is because* f^2 *is of depth*(2). c^2 *in this case must be idempotent. We concluded that if a vertex c in a tree is adjacent* ≥ 2 *leaves, then c will be idempotent. Also, the product of* $d\dot{b}$ *is forced to equal d since d is the only vertex adjacent or including to only d and b. Since this is the complement graph, two vertices are adjacent if and only if their product is non-zero. So instead of a vertex existing z with* $N(x)\bigcup N(y)\subseteq\overline{N(z)}$, $\exists$ *a vertex z which is not adjacent to the same vertices that* $x\bigcup y$ *are not adjacent to. In other words, z is only adjacent to x and/or y.*

Example 6. *By this example, we were able to conclude the following:*



The element a must be idempotent since it is adjacent to ≥ 2 *leaves. The same follows for d. We know that the product* $a\cdot b$ *must be one of the endpoints* $\{b, c, e, f\}$. *Choose an endpoint adjacent to a, say c. We have the following:*

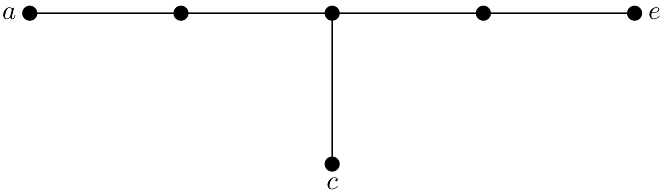
$$a \cdot d = c$$

now multiply both sides by d

$$\begin{gathered} d \cdot a \cdot d = c \cdot d \\ a \cdot d^2 = 0 \\ a \cdot d = 0 \end{gathered}$$

Which is a contradiction. The same would occur if you chose a vertex adjacent to d and multiplied both sides by a. We then conclude that two adjacent vertices cannot both be idempotent or 2-step nilpotent. This would eliminate the possibility for this graph to belong to a zero-divisor semigroup.

Example 7. *The example below is interesting because of the role played by vertex* f.



We know that the end vertices must be two-step nilpotent from various forced conditions including verticies adjacent from leaves being idempotent and not being able to have two idempotent vertices adjacent (which was proved earlier). Since the end vertices are all two-step nilpotent we know that both d and b must be idempotent. Now we need to assign a letter to f^2. *f cannot be idempotent due to contradicting adjacencies. f also cannot be two-step nilpotent due to adjacencies as well (c is two-step nilpotent). So we concluded that f must be transpotent and since the only option for f being transpotent is c, then f^2 must be c.*

Example 8. *The following graph is not the graph of any zero-divisor semigroup.*

One can see from this example that there are no possibilities for the product of $f \cdot c$. The product $f \cdot c$ can only be a vertex that is inclusively adjacent to only f and c. Therefore one can see that you may not have a path that is greater than 4 belonging to a zero-divisor semigroup.

4. Complements of Semigroup Graphs

Definition 11. *Let G be a simple undirected graph. We say G is admissible if G is the complement of a semigroup graph under the zero-divisor rule.*

In this section, G will always be a simple, connected graph, and S will always be a semigroup associated with G. We will denote by v a vertex in G and an element of S at the same time. if $v^2 = 0$, we call v nilpotent (of order 2); unless otherwise specified, nilpotent will always mean nilpotent of order 2. if $v^2 = v$, we call v idempotent. If v is not nilpotent or idempotent, then we call v transpotent. If $d(v^2, v) = n$, we call v n-transpotent.

Proposition 1. *If v is a vertex in an admissible graph G, then v cannot be n-transpotent $\forall n \geq 3$.*

Proof. Suppose $v^2 = x$, and $d(v, x) > 2$. Then $\forall z \in \mathcal{N}(x)$, $d(z, v) > 1$. Hence $zv = 0$, yet $zv^2 = zx \neq 0$. So we have a contradition. □

Thus every element in a semigroup graph can only have one of the four possible potencies: idempotent, nilpotent, 1-transpotent, and 2-transpotent.

Let G be a simple connected graph and let $S \subseteq V(G)$, then x is said to ***bound*** *S if $\forall y \in \mathcal{N}(x)$, $max\{d(y,t) | t \in S\} \leq 1$. The set of boundary vertices of S in G is denoted by $B(S|G)$.*

Definition 12. *The* **depth of** S **in** G *is defined as follows: if* S *consists of a single vertex that has degree 1, then depth(S—G)=0; Otherwise* $depth(S|G) = D(S|G) = min\{d(t,b)|t \in S, b \in B(S|G), t \neq b\}$.

Proposition 2. *Let* $(a,b) \in E(G)$. *Then* $B(\{a,b\}|G) \subseteq B(a|G) \cap B(b|G)$.

Proof. Suppose $x \in B(\{a,b\}|G)$, then $\forall y \in \mathcal{N}(x)$, $d(y,a) \leq 1$ and $d(y,b) \leq 1$, hence $y \in B(a|G)$ and $y \in B(b|G) \Rightarrow y \in B(a|G) \cap B(b|G)$. ☐

Proposition 3. *The depth of any set* S *of more than one vertices in* G *is greater than 0.*

Proof. by definition, we take the minimum of non-reflexive distances, which are all ≥ 1. ☐

Proposition 4. *The depth of* S *in* G *cannot exceed 2.*

Proof. Suppose $D(S|G) > 2$, then $\exists u \in S$, $b \in B(S|G)$, such that $d(u,b) > 2$. By definition of $B(S|G)$, $\forall y \in \mathcal{N}(b)$, $d(y,t) \leq 1$ $\forall t \in S$; in particular, $d(y,u) \leq 1$. But $d(b,u) \geq d(b,y) + d(y,u)$ by the triangle inequality, hence the depth of S cannot be greater than 2. Given a set of vertices S in V(G), if $B(S|G) = \phi$, we call S unbounded. We call S bounded if it is not unbounded. ☐

Theorem 3. *If* G *is admissible, then every edge in* G *must be bounded.*

Proof. We prove that the product of two adjacent vertices in the complement graph must be one of their boundary vertices. Suppose $(a,b) \in E(G)$ and $ab = c \notin B(\{a,b\}|G)$, then $\exists y \in \mathcal{N}(c)$, $d(y,a) > 1$ or $d(y,b) > 1 \Rightarrow ya = 0$ or $yb = 0$ $\Rightarrow yab = 0 \Rightarrow yc = 0$. But since y and c are adjacent, their product cannot be zero, hence we arrived at a contradiction. Next suppose (a,b) is unbounded, then by definition $B(\{a,b\}|G) = \phi \Rightarrow ab$ is undefined. Hence G is not admissible. ☐

Lemma 1. *Suppose* G *is admissible.* $(a,b) \in E(G)$. *Then* $d(ab,a) \leq 2$, *and similarly for* b.

Proof. Suppose $d(ab,a) > 2$. Then $\forall y \in \mathcal{N}(ab)$, $d(y,a) > 1$, otherwiise by the triangle inequality $d(ab,a) \leq d(ab,y) + d(y,a) = 2$, contradicting the assumption that $d(ab,a) > 2$. but then $aby \neq 0$, where as $aby = b(ay) = (b)0 = 0$, a contradiction. ☐

Proposition 5. *Suppose* G *is admissible. If two adjacent vertices* $a, b \in V(G)$ *are both 2-nilpotent, then they must be adjacent to a common vertex and* $D(\{a,b\}|G) = 2$.

Proof. We prove a stronger version of the depth claim. In fact we have $d(ab,a) = 2$ and $d(ab,b) = 2$. Suppose without loss of generality $d(ab,a) \neq 2$, then $d(ab,a) < 2$ by lemma 1, and we have two cases to consider: if $d(ab,a) = 1$, then $(ab)a \neq 0$, but $(ab)a = a^2b = ab = 0$, a contradiction. If $d(ab,a) = 0$, then $d(ab,b) = 1$ and we have a symmetric argument.

Now Since ab must be a boundary vertex of $\{a,b\}$, $D(\{a,b\}|G) \geq 2$. by proposition 1, $D(\{a,b\}|G) \leq 2$, hence $D(\{a,b\}|G) = 2$. Now $\forall y \in \mathcal{N}(ab)$, $d(y,a) \leq 1$ and $d(y,b) \leq 1$. By the triangle inequality, $d(y,a) + d(y,ab) \geq d(a,ab)$ and similarly for b, so we are forced to have $d(y,a) = 1$, $d(y,b) = 1$. Thus y is a vertex adjacent to both a and b, as desired. ☐

Proposition 6. *Let G be admissible. If $(a,b) \in E(G)$ and a is nilpotent, then $ab \notin \mathcal{N}(a)$.*

Proof. Suppose $d(ab,a)=1$. Then since $ab \neq 0$, $(ab)a \neq 0$. But $(ab)a = a^2b = 0$, a contradiction. □

Proposition 7. *Let G be admissible. If $(a,b) \in E(G)$ and a is idempotent, then $ab \in \mathcal{N}(a) \cup \{a\}$.*

Proof. Suppose not, then $(ab)a = 0$. But $(ab)a = a^2b = ab \neq 0$. □

Corollary 1. *Let G be admissible. If $(a,b) \in E(G)$, a nilpotent and b idempotent, then $ab \in \mathcal{N}(b) - \mathcal{N}(a)$*

Proof. This follows immediately from the previous two propositions. □

Proposition 8. *Let G be simple, connected, $|V(G)| \geq 3$, $y \in V(G)$ and let x be a leaf in $\mathcal{N}(y)$. Then $xy \in \mathcal{N}(y) \cap \{z \in V(G) | deg(z) = 1\}$, i.e., xy must be one of the neighboring leaves of y.*

Lemma 2. *If $v \in V(G)$ is not nilpotent, then $v^2 \in B(\{v\}|G)$.*

Proof. Suppose $v^2 = u \notin B(v|G)$. Then $\exists x \in \mathcal{N}(u)$, $d(x,v) \geq 2$. So $xv^2 = (xv)v = 0$. But since u is adjacent to x, $xv^2 = xu \neq 0$, a contradiction. □

5. The Direct Sum of Graphs

Definition 13. *A simple graph Γ is said to be the direct sum of $\{G_i\}_{i\in A}$ if $\forall i \in A$, G_i is a simple connected graph, $V(\Gamma) = \coprod_{i\in A} H_i$, $\forall a \in H_i$, $b \in H_j$, $i \neq j$, $(a,b) \notin E(\Gamma)$, and the induced subgraph of $V(H_i)$ in Γ is isomorphic to G_i itself for each $i \in A$.*

Theorem 4. *Let Γ be the direct sum of $\{G_i\}_{i\in A}$. If $|V(G_i)| \geq 2$ $\forall i \in A$, then Γ is admissible if and only if each G_i is. If $\exists i \in A$, $|V(G_i)| = 1$, then Γ is always admissible.*

Proof. Let $V(\Gamma)$=$\coprod_{i\in A} H_i$, and $\forall i \in A$, the induced subgraph of H_i in P is $\cong G_i$. First suppose $|H_i|$=1 for some $i \in A$. Let $x \in H_i$ abe the only vertex in H_i. $\forall (a,b) \in E(\Gamma)$, define $ab = x$, and let every vertex in Γ be nilpotent. Then it is easy to see that the multiplication rule obeys the adjacency of the graph and every triple product $abc = 0$ $\forall a,b,c \in V(\Gamma)$ not necessarily distinct. Thus the multiplication is automatically associative and commutatively is implicit in the definitions (in fact, the complement of this graph is the refinement of the star graph). Now assume $|H_i| \geq 2$ $\forall i \in A$. If all of G_i's (which we use interchangeably with the induced subgraph of H_i in Γ) are admissible, let $\tilde{a}, \tilde{b} \in G_i$, and let a,b be the isomorphic image of $\tilde{a}\tilde{b}$ in H_i respectively. Let $S(G_i)$ be the associated semigroup of G_i and let $\times_{S(G_i)}$ be its operation. Then define ab=$\tilde{a} \times_{S(G_i)} \tilde{b}$. Since isomorphism is subjective and injective , every pair $a, b \in H_i$, $b \in H_j$, $i \neq j$, define ab=0. These definitions of products are intrinsically commutative, so we just need to check associativity. Given $a,b,c \in V(P)$, up to permutation, these are 3 possibilities:
1.) $a \in H_i$, $b \in H_j$, $c \in H_k$,i,j,k are pairwise distinct.
2.) $a,b \in H_i$, $c \in H_j$, $i \neq j$.
3.) $a,b,c \in H_i$.
In case 1, no matter which two elements you multiply first, the resulting product

is always going to be 0. hence it's trivially associative.
In case 2, suppose we do $(ab)c$. Then since $\tilde{a} \times_{S(G_i)} \tilde{b} \in V(G_i) \cup \{0\}$ by admissibility of G_i. We have $ab \in H_i \cup \{0\}$. Hence $(ab)c$=0. If on the other hand, we multiply a, c first, i.e. $(ac)b$, we have $ac = 0$, hence the product is also 0. All the other orders of multiplication are equivalent to the above two subcases.
In case 3, associativity of the product abc is equivalent to associativity of multiplication in G_i, which we know is true since by assumption, G_i is admissible.

Finally, suppose Γ is admissible. We first prove that $\forall i \in A$, $H_i \cup \{0\}$ must be closed under multiplication of Γ. Let $a, b \in H_i$ for some $i \in A$ and suppose $ab = c \in G_j, j \neq i$. By assumption $|V(G_j)| \geq 2$, hence $N(c) \neq \emptyset$, since G_j is also connected by definition of direct sum. Let $d \in N(c)$, then $cd \neq 0$. But $cd = abd = 0$, a contradiction. Next $\forall a, b \in H_i$, define a multiplication $\times_{S(G_i)}$ in $V(G_i) \cong H_i$ by the following rule: $\tilde{a} \times_{S(G_i)} \tilde{b} = ab$, when the multiplication on the right is the multiplication in Γ. Then first of all $V(G_i) \cup \{0\}$ is closed under $\times_{S(G_i)}$ since $H_i \cup \{0\}$ is under the multiplication of Γ. Furthermore, isomorphism preserves commutativity and associativity. Hence G_i is admissible, with the associated semigroup $S(G_i)$=$(V(G_i), \times_{S(G_i)})$. □

6. Admissibility of Trees

Recall the following definition from the previous section:

Definition 14. *Let G be a tree, and let L denote the set of degree 1 vertices in G. Define the tree depth of a vertex $a \in V(G)$ to be $Tdepth(a|G) = min\{d(a, z)|z \in L\}$. When context is clear, we also abbreviate $Tdepth(a|G)$ as just $Tdepth(a)$.*

We show the definition of depth (of a single vertex) for trees is consistent with the general definition of depth when $depth(a|G) \leq 2$.

Theorem 5. *Let G be a tree and let $a \in V(G)$. Then $Tdepth(a|G) = depth(a|G)$ provided that one of them is ≤ 2.*

Before proving the theorem, we introduce several technical lemmas.

Lemma 3. *If G is connected and $x \in V(G)$, then $\forall y \in V(G) - \{x\}$, $\exists z \in \mathcal{N}(x)$, such that y,z are connected in $G - \{x\}$.*

Proof. Suppose not, then $\exists y \in V(G) - \{x\}$, such that $\forall z \in \mathcal{N}(x)$, y, z are not in the same connected component in $G - \{x\}$. But y,x are connected in G, so $y = x$(must), a contradiction. □

Lemma 4. *Let G be a simple connected graph. Then $\forall v \in V(G)$, $v \in B(v|G)$.*

Proof. By definition of neighborhood, $\forall x \in \mathcal{N}(v)$, $d(x, v) = 1 \leq 1$. Hence v is a boundary vertex of itself in G. □

Lemma 5. *For all $v \in V(G)$, $B(v|G) \subseteq Z \cup \{v\}$.*

Proof. By the previous lemma, $v \in B(v|G)$. Suppose $x \in B(v|G)$ and $x \notin Z \cup \{v\}$. Then $v \in V(G) - \{x\}$ and $deg(x) \geq 2$, hence $\mathcal{N}(x) = \{\alpha_i\}_{i \in A}$, where $|A| \geq 2$. Since G is a tree, we also have that x is a cut-vertex. Now by the assumption that v is a boundary vertex of $\{x\}$, $d(\alpha_i, v) \leq 1$ $\forall i \in A$. So in particular there is a path from v to any α_i in $G - \{x\}$, partly because $v \in V(G) - \{x\}$: the path is either one-step, or degenerate. Now using the previous lemma, we see that since $\forall w \in V(G) - \{x\}$, w is connected to at least one α_i, and all α_i are connected to v, we have $G - \{x\}$ must be connected, contradicting the fact that x is a cut vertex. □

Proof. (of the theorem) We go through each of the three possible depths. When the tree depth of $x \in V(G)$ is 0. x must be a degree 1 vertex. By definition the depth of a degree 1 vertex is always 0, and only degree 1 vertices have depth 0 by observation 2. When $Tdepth(x) = 1$, $\exists y \in N(x)$, $deg(y)$=1. We claim that $y \in B(x|G)$. Next, suppose $deg(x) = 1$, then $Tdepth(x)$=0$\neq$ 1 so $deg(x) > 1$ $\Rightarrow depth(x) > 0$ by observation 2. Hence $depth(x) = min\{d(x,t)| + B(x|G) = 1$ since $d(x,y) = 1$ and $t \in B(x|G)$. Conversely, if $x \in V(G)$ and $depth(x)$=1, then $deg(x) > 1$, and $\exists\ y \in B(x|G)$, $d(x,y) = 1$. But since $B(x|G)$, let $\{x,y,z\}$ be the path of length 2 from x to y. Then $d(y,z)$=1, so $N(y)$=$\{z\}$. But then $d(x,z)$=1, so $y \in B(x|G)$. Next we want to see that $u \in V(G)$, $deg(y)$=1, and $d(x,y) <$ 2. Suppose there exists such u, then by definition of tree depth, $Tdepth(x) <$ 2, contrary to the assumption so $\forall\ u \in \{V(G)\}|deg(t) = 1\}d(x,u) \geq 2$. Since $B(x|G)$$\subset$$\{t \in V(G)|deg(t) = 1\}$, we have $depth(x|G) \geq 1$. But from $d(y,x) = 2$, we know $depth(x|G) \leq 2$ hence $depth(x|G)$=2. Conversely, if $depth(x|G) = 2$, then $min\{d(x,t)|t \in B(x|G)\}$=2. Let $y \in B(x|G)$ and $d(x,y) = 2$. Since $B(x|G)$ $\subset$$\{t \in V(G), deg(t) = 2\}$, we have $deg(y) = 1$$\Rightarrow$ $Tdepth(x) \leq 2$. So we just need to check that $Tdepth(x) \notin$$\{$0,1$\}$. $Tdepth(x) \neq 0$ because if it is equal, then s will be a degree 1 vertex, which in turn will have normal depth 1. Suppose $Tdepth(x) = 1$, then $\exists\ x \in V(G)$, $deg(z) = 1$, and $d(x,z)$=0$\leq$ 1, $x \in B(x|G)$. Thus $depth(x|G) \leq 1$, contrary to assumption. Thus a vertex in G a tree is unbounded if and only if tree depth must be greater than 2, since the tree depth of a vertex in G is always defined.

□

Lemma 6. *Let G be an admissible tree and let $u, v \in V(G)$, $u \neq v$. Then $uv \in L$.*

Proof. By previous lemma, $uv \in B(\{u,v\}|G)$. Also $B(\{u,v\}|G) \subseteq B(u|G) \cap B(v|G)$. Next $B(u|G) \subseteq L \cap \{u\}$, $B(v|G) \subseteq L \cap \{v\}$ by previous lemma. We want to show that $B(u|G) \cap B(v|G) \subseteq L$. First suppose both $u, v \in L$, then $B(u|G) \cap B(v|G) \subseteq L \cup \{u,v\} = L$. Next suppose without loss of generality that $u \in L$ whereas $v \notin L$. Then $v \notin B(u|G)$ since u, v are distinct. So $B(u|G) \cap B(v|G) \subseteq L \cup \{u\} = L$ since by assumption $u \in L$. Finally if both $u, v \notin L$, then $u, v \notin B(u|G) \cap B(v|G)$ because $u \notin B(v|G)$ and $v \notin B(u|G)$. Hence $B(u|G) \cap B(v|G) \subseteq L \cup \{u\} \cup \{v\} - \{u,v\} = L$. So in all three cases, we have $B(u|G) \cap B(v|G) \subseteq L \Rightarrow uv \in L$. □

Lemma 7. *Let G be a tree. If $x \in V(G)$ and $Tdepth(x) \geq 3$, then G is not admissible.*

Proof. Let $y \in \mathcal{N}(x)$. We claim that (x,y) is unbounded in G. Suppose it is bounded. Then $\exists z \in B(\{x,y\}|G)$. And $\exists w \in \mathcal{N}(z)$, $d(w,x) \leq 1$, $d(w,y) \leq 1$, since G cannot be a singleton by virtue of the fact that there are vertices in it that have tree depth nonzero. Also from previous lemma, we know that $B(\{x,y\}|G) \subseteq B(x|G) \cap B(y|G)$. Also since $B(x|G), B(y|G) \subseteq \{t \in V(G) : deg(t) = 1\}$, we have $B(x|G) \cap B(y|G) \subseteq \{t \in V(G) : deg(t) = 1\}$. Hence $B(\{x,y\}|G) \subseteq \{t \in V(G) : deg(t) = 1\}$. But then $deg(z) = 1$ and $d(x,z) \leq d(x,y) + d(y,z) = 1 + 1 = 2$, contrary to the depth 3 assumption. □

Lemma 8. *If G is an admissible tree, then G cannot have 2 adjacent depth 2 vertices.*

Proof. Suppose $u, v \in V(G)$, and $depth(u) = depth(v) = 2$. Then we claim the edge (u,v) is unbounded in G. Suppose it is bounded, then $B(\{u, v\}|G) \neq \phi$. So let $x \in B(\{u, v\}|G)$, and let $y \in \mathcal{N}(x)$. Then $d(y, u) \leq 1$, $d(y, v) \leq 1$. If $d(y, v) = d(y, u) = 1$, we have a common adjacent vertex to both u and v, which forms a cycle, contrary to the assumption that G is a tree. So either $d(y, u) = 0$ or $d(y, v) = 0$. Without loss of generality, assume $d(y, u) = 0$. Then y=u, and $x \in \mathcal{N}(y)$ is a boundary vertex of (u,v), hence a boundary vertex of both $\{u\}$ and $\{v\}$, which means that $deg(x) = 1$. But then $d(x, y) = 1 \Rightarrow depth(y) = Tdepth(y) = 1$, contrary to the assumption. ☐

Corollary 2. *Let G be an admissible tree and let $x \in V(G)$. If $\exists y \in \mathcal{N}(x)$, $depth(y) = 2$, then $depth(x) = 1$.*

Proof. If $depth(x) \neq 1$, then either $depth(x) > 1$ or $depth(x) = 0$. By the above lemma, $depth(x)$ cannot be 2, so $depth(x) \geq 3$. But by another lemma above, this will force G to be not admissible. So we are only left with the choice that $depth(x) = 0$. But this will imply that $depth(y) \leq 1$, contradicting the assumption that $depth(y) = 2$. ☐

Lemma 9. *Let G be a simple, connected graph and let $x, y \in V(G)$. If $d(x, y) = 2$, then $\exists w \in V(G)$, $d(x, w) = d(w, y) = 1$.*

Proof. ☐

Lemma 10. *Suppose G is an admissible tree and $x \in V(G)$ with $depth(x) = 2$. Then x must be nilpotent.*

Proof. By one of the previous lemmas, $\forall y \in \mathcal{N}(x)$, $depth(y) \neq 2$. And we also know that $\forall t \in \mathcal{N}(x)$, $depth(y) \leq 1$. But if $\exists y \in \mathcal{N}(x)$, $depth(y) = 0$. Then $y \in L \Rightarrow depth(x) = Tdepth(x) \leq d(x, y) = 1$, contrary to the assumption that $depth(x) = 2$. So $\forall y \in \mathcal{N}(x)$, $depth(y) = 1$. Next we prove that x cannot be idempotent, 1-transpotent, or 2-transpotent, hence must be nilpotent. Suppose x is idempotent. First of all, $xy \in B(\{x, y\}|G) \subseteq L$. Hence $d(xy, x) \geq 2$. But then $x^2y = 0$ whereas $x^2y = xy \neq 0$ since x is idempotent. Thus we have a contradiction. Next, x cannot be 1-transpotent because $\nexists z \in \mathcal{N}(x)$, such that $z \in B(x|G) = \{x\} \cup L$. Finally,, if x is 2-transpotent, let $x^2 = c$. Then $c \in L$. Let $a \in \mathcal{N}(c) \cap \mathcal{N}(x)$, then $d(a, c) = 1$, hence $ac \neq 0$; similarly $ax \neq 0$. but since $a \neq x$, $ax \in L$. Thus $d(ax, x) \geq 2 \Rightarrow (ax)x = 0$, contradicting the fact that $(ax)x = ax^2 = ac \neq 0$. Thus x must be nilpotent.

☐

Lemma 11. *Let G be an admissible tree, and let $x \in V(G)$ be adjacent to at least 2 leaves, i.e., $|\mathcal{N}(x) \cap L| \geq 2$. Then x must be idempotent.*

Proof. By the condition, $|\mathcal{N}(x)| \geq 2$, hence $|V(G)| \geq 3$. So we can use the lemma which states that the product of x and one of its leafy neighbor must be one of the leafy neighbors of x. Suppose $u, v \in \mathcal{N}(x) \cap L$, then we have $xu \in \mathcal{N}(x) \cap L$ and $xv \in \mathcal{N}(x) \cap L$. But then $x^2u = x(xu) \in \mathcal{N}(x) \cap L$ and $x^2v = x(xv) \in \mathcal{N}(x) \cap L$. So in particular, $x^2u, x^2v \neq 0$. So $x^2 \in (\mathcal{N}(u) \cup \{u\}) \cap (\mathcal{N}(v) \cup \{v\}) = \{x\}$ $\Rightarrow x^2 = x$. ☐

Lemma 12. *Let G be an admissible tree, and let $x \in V(G)$ be adjacent to a depth-2 vertex y. Then x must be idempotent.*

Proof. We show that x cannot be nilpotent, 1-transpotent, or 2-transpotent, thus must be idempotent. Suppose first that x is nilpotent. Then by a proposition before, there must be a vertex adjacent to both x and y, since y by the previous lemma must also be nilpotent. But then we have a cycle, which cannot exist in a tree. Suppose now that x is 1-transpotent, then $x^2 = t$ for some $t \in L \cap \mathcal{N}(x)$, since $x^2 \in B(x|G)$ and $B(x|G) \subseteq L \cup \{x\}$, by our previous result. $\square$

Proposition 9. *Let G be an admissible tree. If $x \in V(G)$ has depth 2 and $y \in \mathcal{N}(x)$, then $xy \in \mathcal{N}(y) \cap L$.*

Proof. From previous lemma, we know that $xy \in L$ provided that $x \neq y$. Suppose $xy = z \in L - \mathcal{N}(y)$, we show that either $d(z,x) \geq 3$ or $d(z,y) \geq 3$. First of all, by previous result, $Tdepth(y) = 1$, $y \notin L \Rightarrow d(z,y) \neq 0$. Also $z \in L$ and yet $z \notin L \cap \mathcal{N}(y)$. So $d(z,y) \neq 1$. Finally if $d(z,y) = 2$, then $d(x,z) \neq 0$ because $x \notin L$; $d(x,z) \neq 1$ because $\forall t \in \mathcal{N}(x)$, $depth(t) = 1$ which implies that $t \notin L$. So if $d(z,x) < 3$ $d(z,x)$ must be $= 2$.Suppose $d(z,x) = 2$. then $\exists u \in V(G)$, such that $d(u,x) = d(u,z) = 1$ and $\exists v \in V(G)$ such that $d(v,y) = d(v,z) = 1$. If $u = v$, then $\{u,x,y\}$ would form a 3-cycle, not allowed in G a tree. So $u \neq v$, but then $\{x,y,v,z,u\}$ would form a 5-cycle, still not allowed in G. Hence $d(z,x) \geq 3$. Thus we have either $d(z,x) \geq 3$ or $d(z,y) \geq 3$. But then, $z \notin B(\{x,y\}|G)$ since $\forall t \in \mathcal{N}(z)$, either $d(t,x) \geq 2$ or $d(t,y) \geq 2$. So by one of the previous lemmas, $xy \neq z$. $\square$

Lemma 13. *(Distance Lemma) Let G be any connected admissible graph and suppose $u,v \in V(G)$ are adjacent, then $d(uv,u) \geq 2$ and $d(uv,v) \leq 2$.*

Proof. By previous lemma, $uv \in B(\{u,v\}|G)$, and by definition of boundary vertex, $\forall t \in \mathcal{N}(uv)$, $d(t,u) \leq 1$ and $d(t,v) \leq 1$. Since G is connected, and $\{u,v\} \subseteq V(G)$, we have $|V(G)| \geq 2$ and $\mathcal{N}(uv) \neq \phi$. Let $t \in (N)(uv)$, then $d(t,uv) = 1$, so $d(uv,u) \leq d(uv,t) + d(t,u) \leq 1 + 1 = 2$ and similarly $d(uv,v) \leq 2$. $\square$

Lemma 14. *(Inequality Lemma) Let G be an admissible tree, and let $u,v \in V(G)$ be adjacent. Then $d(uv,v) \neq d(uv,u)$.*

Proof. Since $d(uv,u) \leq 2$ and $d(uv,v) \leq 2$ by distance lemma, we only need to check that the following three cases cannot happen:
1. $d(uv,u) = d((uv,v) = 0$;
2. $d(uv,u) = d((uv,v) = 1$;
3. $d(uv,u) = d((uv,v) = 2$.
Case 1 is obviously impossible since $u \neq v$. If case 2 is true, then $\{u,v,uv\}$ forms a 3-cycle, contrary to G being a tree. If case 3 is true, then by the intermediate vertex lemma, $\exists a \in V(G)$ such that $d(uv,a) = 1$, $d(a,u) = 1$, and $\exists b \in V(G)$ such that $d(uv,b) = 1$, $d(b,v) = 1$. First of all, if $a = b$, then $d(u,a) = d(v,a) = d(u,v) = 1$. Furthermore, $u \neq a$ since otherwise $d(uv,u) = 1$, contrary to assumption, and similarly $v \neq a$; $u \neq v$ by assumption of adjacency. So $\{u,v,a\}$ would form a 3-cycle, not allowed in trees. Thus we may assume that $a \neq b$, but then $d(uv,a) = d(a,u) = d(u,v) = d(v,b) = d(b,uv) = 1$. To show that a,b,u,v,uv are all distinct, we divide them into 3 groups: $\{u,v\}$, $\{a,b\}$, and $\{uv\}$. Notice that the elements within each group are distinct by assumption; next without loss of generality, we

show that $u \neq a$, $b \neq uv$ and $v \neq uv$.But those follow immediately from the distance between those vertices, hence $\{u, a, uv, b, v\}$ forms a 5-cycle, not allowed in trees. □

Corollary 3. *Let $u, v \in V(G)$ be adjacent depth 1 vertices, then $uv \in (\mathcal{N}(u) \cup \mathcal{N}(v)) \cap L$.*

Proof. Since $uv \in L$, and $u, v \notin L$ by depth 1 assumption, we know that $d(uv, u) \geq 1$ and $d(uv, v) \geq 1$. But by the inequality lemma, it cannot happen that $d(uv, u) = d(uv, v) = 1$ or $d(uv, u) = d(uv, v) = 2$. And we also know that $d(uv, u) \leq 2$ and $d(uv, v) \leq 2$ from distance lemma. So we are left with the options $d(uv, u) = 1$, $d(uv, v) = 2$, or $d(uv, u) = 2$, $d(uv, v) = 1$. In the first case, $uv \in \mathcal{N}(u)$, whereas in the second case, $uv \in \mathcal{N}(v)$. Hence $uv \in \mathcal{N}(u) \cup \mathcal{N}(v) \Rightarrow uv \in (\mathcal{N}(u) \cup \mathcal{N}(v)) \cap L$. □

Theorem 6. *Let G be an admissible tree. If $u, v \in V(G)$ are adjacent, then u, v cannot be both idempotent.*

Proof. Suppose on the contrary that $u^2 = u$ and $v^2 = v$. By Corollary above ,$uv \in (\mathcal{N}(u) \cup \mathcal{N}(v)) \cap L$. Since $\mathcal{N}(u) \cap \mathcal{N}(v) = \phi$ (otherwise we have a 3-cycle), we may assume without loss of generality that $uv \in \mathcal{N}(u) \cap L$ and $uv \notin \mathcal{N}(v)$. Then $(uv)v = 0$ since uv and v are not adjacent in G. But $(uv)v = uv^2 = uv \neq 0$, a contradiction. □

7. Graph Separability

Definition 15. *Let S be an induced subgraph of G, let v be a vertex in S. If we attach an infinite path to v which results in an extended induced graph S', then $depth(v|S')$ is called the depth of v in G relative to S, denoted by $D(v|G)rel.S$.*

Definition 16. *Let C be an induced subgraph of G, and let $x \in C$. Let B be the induced subgraph of G with vertices $V(G) - (V(C) - \{x\})$, and let T_x be the connected component of B containing x. then we denote $T_x - \{x\}$ by $Fol(x|C)$, or the foliage of x in G with respect to C.*

Definition 17. *A graph G is said to be separable by S if every vertex of G not in S belongs to the foliage of some vertex in S and $\forall u, v \in V(S)$, $u \neq v$, $Fol(u) \cap Fol(v) = \phi$. A graph is said to be strongly separable by S if it is separable by S and each vertex in S has nonempty foliage.*

Theorem 7. *Let G be a graph strongly separable by Z, and let Fol(v) be the foliage of v with respect to Z $\forall v \in Z$. If G is admissible, then $\forall v \in Z$, v is relatively bounded in $Fol(v) \cup \{v\}$. For all $v \in Z$, $D(v|G)$ rel. $Fol(v) \cup \{v\} = 2$, v must be 2-nilpotent. And G must also satisfy all of the following conditions:*

1. *$\nexists u, v \in Z$ adjacent, such that both u and v are 2-nilpotent.*
2. *$\forall v \in Z$, both $Fol(v)$ and $Fol(v) \cup \{v\}$ are admissible.*
3. *If $D(v|G)rel.(Fol(v) \cup \{v\}) = 2$, then v must be nilpotent.*

Theorem 8. *If G is connected, and if it is separable by Z, then Z must be connected also.*

Proof. Suppose Z is not connected, then $Z = C_1 \cup C_2$, where C_1 and C_2 are disjoint and nonempty. Let $x \in C_1$ and $y \in C_2$, and let $P = \{v_0, v_1, \ldots, v_k\}$ be a path from x to y, where $v_0 = x$ and $v_k = y$. Let $\alpha = max\{i|v_i \in C_1\}$ and $\beta = max\{i|v_i \notin C_2\}$.

Then $\alpha < \beta$ because $\{i|v_i \in C_1\} \subset \{i|v_i \notin C_2\}$ and $\alpha \neq \beta$ since otherwise v_α and $v_{\beta+1}$ will be adjacent, contradicting the fact that C_1 and C_2 are disjoint. But then the set of vertices $\{v_{\alpha+1}, v_{\alpha+2}, \ldots, v_\beta\}$ form a nonempty subpath of P that's not contained in C_1 or C_2, hence $\{v_{\alpha+1}, v_{\alpha+2}, \ldots, v_\beta\} \nsubseteq C_1 \cup C_2 = Z$ and it is connected to $v_\alpha \in C_1$ and $v_{\beta+1} \in C_2$, hence by definition, $\{v_{\alpha+1}, v_{\alpha+2}, \ldots, v_\beta\} \subseteq Fol(v_\alpha) \cap Fol(v_{\beta+1})$. So G is not separable by Z. ☐

Conjecture 2. *Let G be strongly separable by Z with at least two vertices,* $\forall x \in Z$, $D(x|G)rel.(Fol(x) \cup \{x\}) = 2$, *label x nilpotent. Then G is not admissible if any of the following conditions hold:*

1. *There is a vertex in Z whose relative depth in its foliage with respect to Z is great than 2.*
2. *There are two adjacent nilpotent vertices in Z;*
3. *The rest of the vertices in Z do not satisfy the alternating distribution property.*

Proof. We give a partial proof of the conjecture. First we prove that $\forall x \in Z$, $Fol(x|Z) \cup \{x, 0\}$ is closed under the multiplication of the original semigroup. Suppose there exists $x \in Z$ such that $Fol(x|Z) \cup \{x\}$ is not closed under multiplication, then we have three types of products to consider: x^2, xy, where $y \in Fol(x)$ and wy, where $w, y \in Fol(x)$. Also each of the above three types of products can take on the value of two different kinds of vertices outside $Fol(x) \cup \{x\}$, i.e., a different vertex in Z or a vertex in the foliage of a different vertex in Z. So we consider each case separately. 1.1 If $x^2 = c$, where $c \in Z$ and $c \neq x$, then since y has nonempty foliage, $\exists a \in Fol(c)$, $ca \neq 0$. Hence $x^2a \neq 0$. But by definition of separability, $xa = 0$. Thus we arrived at a contradiction. 1.2 If $x^2 = a$, where $a \in Fol(c)$ and $c \in Z$, $c \neq x$, then $\exists b \in Fol(c) \cup \{c\}$, $ba \neq 0$, since b is either connected to y, or if it is not, must be connected to something in the foliage of c because the graph is all connected. But $ba = bx^2 = 0x = 0$, by separability. 2.1 Let $y \in Fol(x)$, if $yx = c \in Z$, $c \neq x$, then $\exists a \in Fol(c)$, $ca \neq 0$. But $ca = yxa = x(ya) = x0 = 0$. 2.2 If $yx = a \in Fol(c)$, where $c \in Z$, and $c \neq x$, then $\exists b \in Fol(c) \cup \{c\}$, $ba \neq 0$. But $ba = b(yx) = (by)x = 0$. 3.1 Let both $y, w \in Fol(x)$, where y and w are not necessarily distinct. Then if $yw = c \in Z$,$c \neq x$, then $\exists a \in Fol(c)$, $ca \neq 0$. But $ywa = 0$. 3.2 If $yw = a \in Fol(c)$, where $c \in Z$, and $c \neq x$, then $\exists b \in Fol(c) \cup \{c\}$, $ba \neq 0$. But $ba = b(yw) = (by)w = 0$, again by separability. Hence we have closure of $Fol(x|Z) \cup \{x, 0\}$. Next we also need a stronger closure condition, i.e., $Fol(x|Z) \cup 0$ is also closed under multiplication of the semigroup. $\forall y, w \in Fol(x)$, their product can be x, a different vertex from x in Z, or a vertex in the foliage of a different vertex from x in Z. The last two cases are already considered in 3.1 and 3.2, so we are only left with the first case. Suppose $yw = x$, then Next we prove that there cannot be vertex in Z whose relative depth in its foliage is great than 2, i.e., infinite. This means that $\exists x \in Z$, such that if S is the graph obtained by attaching an infinite-length branch to x in the induced subgraph in G of $\{x\} \cup Fol(x)$, then x has no boundary vertex in S. Since $Fol(x) \neq \phi$, $\exists y \in Fol(x)$, $xy \neq 0$.

The rest of the proof will follow in a later work. ☐

8. Zero-divisor Semigroups on Triagulated Compact Surfaces

Given a compact surface S, let $C(T_i)$ be the class of triangulations of S such that each T_i is the graph of some zero-divisor semigroup. We identify the elements of $C(T_i)$ and explore their properties.

Definition 18. *A surface S is said to be **compact** if and only if S is both bounded and closed.*

Definition 19. *A **triangulation** of a surface S is a cell decomposition of S such that every face has exactly three edges, no face meets itself, and any two faces meet along a single edge with the ends meeting either at a single vertex or not at all.*

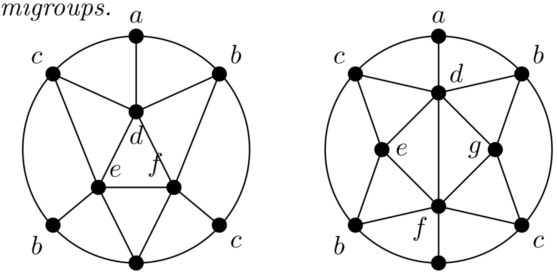
Definition 20. *A cycle C is said to be $non-contractible$ if the topological loop generated by C is not contractible.*

Definition 21. *The $edge-width$ of an embedded graph G is defined as the length of a shortest non-contractible cycle in G.*

Definition 22. *If $k \geq 3$ is an integer, a triangulation T of a surface S is a $k-minimal$ triangulation if the edge-width of T is k and each edge is contained in a non-contractible k-cycle.*

Remark. *The class of k-minimal triangulations of a surface is finite for any k and any surface.*[5]

Theorem 9. *The 3-minimal triangulations of the projective plane are all graphs of zero-divisor semigroups.*



(fig. 1) (fig. 2)
3-minimal triangulations on the projective plane

Proof. The zero-divisor semigroup associated to a complete graph is the trivial semigroup. The graph in Fig. 1 is a complete graph on six vertices so the associated semigroup is indeed the trivial semigroup presented by the multiplication table below.

*	a	b	c	d	e	f
a	0	0	0	0	0	0
b	0	0	0	0	0	0
c	0	0	0	0	0	0
d	0	0	0	0	0	0
e	0	0	0	0	0	0
f	0	0	0	0	0	0

zero-divisor semigroup of fig.1

In Fig. 2, vertex d is adjacent to every other vertex. Thus the associated zero-divisor semigroup can be described as below.

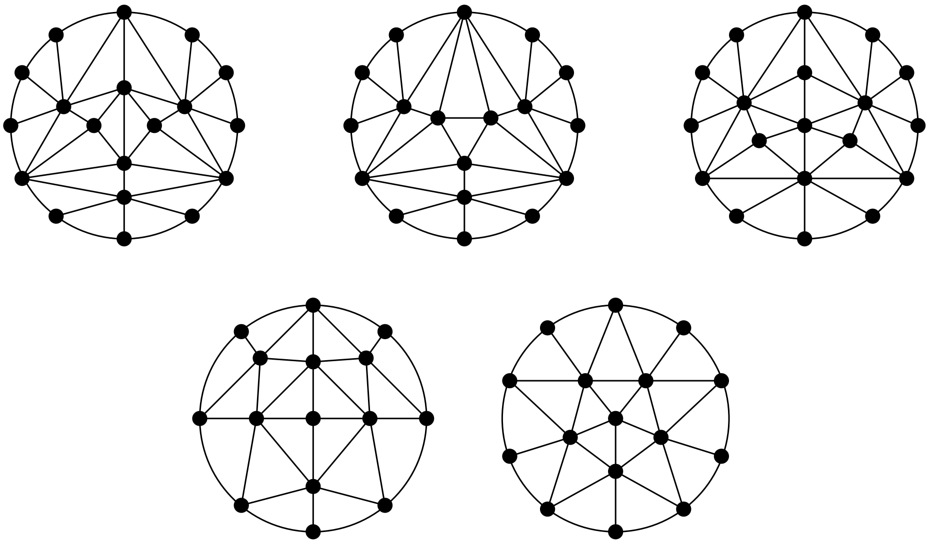
textit

*	a	b	c	d	e	f	g
a	d	0	0	0	d	0	d
b	0	d	0	0	0	0	0
c	0	0	d	0	0	0	0
d	0	0	0	0	0	0	0
e	d	0	0	0	d	0	d
f	0	0	0	0	0	d	0
g	d	0	0	0	d	0	d

zero-divisor semigroup of fig. 2

□

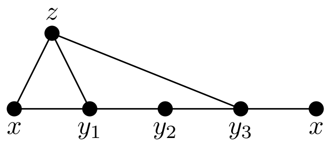
Theorem 10. *None of the 4-minimal triangulations of the projective plane are graphs of zero-divisor semigroups.*



4-minimal triangulations of the projective plan

Theorem 11. *Given a triangulation T of a surface, if T represents a zero-divisor semigroup, then T has an edge-width=3.*

Proof. Let T be a triangulation of a surface S. Suppose that T has an edge-width of 4 and T is a zero-divisor semigroup graph. Let C be a cycle of length 4 in T such that x, y_1, y_2, $y_3 \in C$ and (x, y_1), (y_1, y_2), (y_2, y_3), (y_3, x) are edges in C. Because T is a triangulation, we know that the edge $x \cdot y_1$ is the edge of exactly two triangles. Consider one triangle $x - y_1 - z$. Because z is adjacent to x, z must also be adjacent to y_3. So the cycle $x - z - y_3 - x$ is a cycle of length three. This is a contradiction, so T must have an edge-width of 3.



Let T be a triangulation of a surface S. Suppose that T has an edge-width of n where n is some positive integer greater than 3 and T is a representation of a zero-divisor semigroup. Let C be a cycle of length n in T such that x, y_1, y_2, …, $y_{n-1} \in C$ and (x, y_1), (y_1, y_2), (y_2, y_3), …, (y_{n-2}, y_{n-1}), (y_{n-1}, x) are edges in C. Because T is a triangulation, we know that the edge $x \cdot y_1$ is the edge of exactly two triangles. Consider one triangle $x - y_1 - z$. Because z is adjacent to x, z must also be adjacent to y_{n-1}. So the cycle $x - z - y_{n-1} - x$ is a cycle of length three. This is a contradiction, so T must have an edge-width of 3. Any triangulation that is a zero-divisor semigroup graph must have an edge-width equal to three.

□

Definition 23. *An* ***edge contraction*** *is defined by taking two vertices* v' *and* v'' *joined by an edge and replacing them by a single vertex* v *where all edges meeting* v' *and* v'' *meet* v *and all double edges are replaced by single edges.*

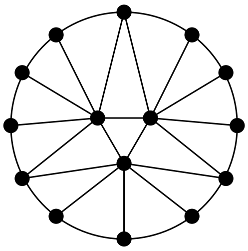
Remark. *The inverse of an edge contraction is called a* ***vertex split****.*

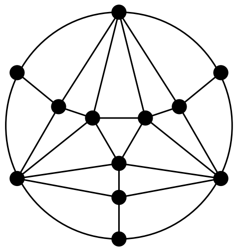
Remark. *Given a triangulation* T*, we say that an edge* e *can be contracted if the resulting graph is a triangulation.*

On a surface S, every triangulation of S can be found from the 3-minimal triangulations by performing a series of vertex splits and edge contractions. Using vertex splits and edge contractions on 3-minimal and 4-minimal triangulations we can develop the entire range of triangulations which have an edge-width equal to three and, further, discover the class of triangulations associated to zero-divisor semigroups on S and explore their properties.

Example 9. *In fig. 1, vertex* b *can be split an infinite number of times and the resulting graph will always be a zero-divisor semigroup graph. This gives an infinite sequence of triangulations wich are all zero-divisor semigroup graphs.*

Example 10. *Below are examples of boundary graphs. If any single contractible edge is contracted then the resulting graph is a zero-divisor semigroup graph.*





9. Clique Homology

We investigate clique homology on zero-divisor semigroup graphs G_z. Clique homology was first introduced in [DD]. The simplicies for clique homology are complete graphs. In the semigropu, an n-simplex is a set k=$\{a_1, a_2, ..., a_{n+1}\}\in S\|a_i \cdot a_j$=$0$,$i \neq j$,$1 \leq i$,$j \leq n+1\|$.

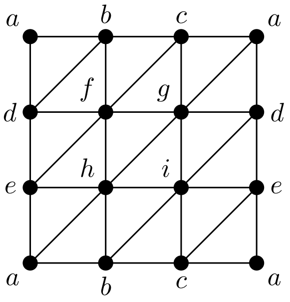
Denote the group of k-chains by $S_k(G_z)$. We have the sequence

$$...\rightarrow S_3(G_z) \overset{\delta_3}{\rightarrow} S_2(G_z) \overset{\delta_2}{\rightarrow} S_1(G_z) \overset{\delta_1}{\rightarrow} S_0(G_z)\rightarrow 0,$$

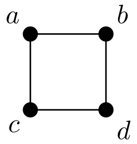
The i^{th} Homology group is defined by $H_i(G_z)$=$kernel(\delta_i)/image(\delta_{i+1})$.

We primarily looked at H_1 in which we took the *image* of the 2-simplex and the *kernel* of the 1-simplex.

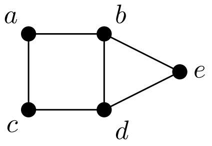
For the following triangulation of the torus, we computed $H_1(G_z)$=$\{0\}$. This is interesting because the *Simplicial Homology* of the torus is $\mathbb{Z}$.



We then continued to compute the H_1 of the following 4-cycle which was $\mathbb{Z}$.



Then we added a triangle to the outside face by adding two-edges and a vertex to one of the original edges. i.e. in this case we added vertex e to edge bd as shown below.



This also gave us a H_1 of $\mathbb{Z}$.
Which led us to the following theorem.

Theorem 12. *Given the Clique Homology of a graph G_z. Adding one vertex and two edges to an original edge to create one additional triangle will not effect the Clique Homology of G_z.*

Proof. Let G_z be a graph with give $H_1(G_z)$.

The map δ_1 is given be a matrix in which the number of columns represent the number of edges and the number of rows represents the number of vertices. In order to obtain an additional triangle, we add one vertex v to an edge in the original graph. The $Kernel$ matrix in this case δ_1 then increases dimension from $m \times n$ to $m + 1 \times n + 2$. There will be a 1 and -1 in the $x_{m+1 \times n+1}$ entry and the $x_{m+1 \times n+2}$ entry. This increases the dimension of nullspace by one.

Now look at the rank of δ_2. The map δ_2 is given by a matrix in which the number of columns corresponds to the number of K_3 graphs or triangles, whereas the number of rows corresponds to the number of edges. Therefore, when adding the triangle to the original graph, the matrix for δ_2 increases from $m \times n$ to $m+2 \times n+1$. There will be a 1 and -1 in the $x_{m+1 \times n+1}$ entry and the $x_{m+2 \times n+1}$ entry. Meaning there will be a 1 and -1 in the same added column. Columns represent triangles and since there are three edges in a triangle there must be an additional 1 or -1 in the added column. When reduced to row echelon form, we see the rank of δ_2 increases by one. Therefore $H_1(G_z)$ is unchanged. ☐

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