AN UPPER BOUND ON THE PEBBLING NUMBER FOR GRAPHS AND A LOOK AT THE 2-PEBBLING PROPERTY

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Abstract. Given a graph G on n vertices, we prove upper bounds on the pebbling number. For any graph of radius r, $f(G) \leq (\frac{2^r-1}{r})n + O(1)$, and for a large class of graphs of diameter d, $f(G) \leq (\frac{2^{\lceil \frac{d}{2} \rceil}-1}{\lceil \frac{d}{2} \rceil})n + O(1)$.

We also prove upper bounds for the (r, m) pebbling number of paths, trees, cycles, and most general graphs.

Lastly, we take a look at the 2-Pebbling Property and show that it holds for certain classes of graphs.

1. Preliminaries

Throughout this work, $G = (V, E)$ will denote a connected graph on n vertices. Suppose pebbles are placed on the vertices of G according to a *distribution* D given by $D : V \to \mathbb{N} \cup \{0\}$. If a vertex u has two or more pebbles, then we can make a *pebbling move*, which consists of the removal of two pebbles from u and placing one of these pebbles on any adjacent vertex v.

Given a target vertex set $t \subseteq V(G)$, the *pebbling number* of G with target t, $f(G, t)$, is the least number of pebbles such that for any distribution of these pebbles, a pebble can be placed onto at least one vertex in t after some number (possibly 0) of pebbling moves.

We say that a vertex set t is *pebblable* by the vertex set A, where $t, A \subseteq V(G)$, if given a distribution D^*,

$$D^*(v) = \begin{cases} D(v), & \text{if } v \in A; \\ 0, & \text{otherwise.} \end{cases}$$

a pebble can be moved to a vertex of t after some number (possibly 0) of pebbling moves on D^*. For a distribution D and $v \in V(G)$, we denote the *potential* of v, $p(v)$, as the greatest number of pebbles that can be moved to v after some sequence of pebbling moves on distribution D.

A *maximal saturation distribution* of pebbles refers to any distribution of the maximum number of pebbles such that G is not pebbleable. As a result, the number of pebbles in a maximum saturation distribution is one less than the pebbling number.

In this paper, we prove that the pebbling number of a large class of graphs of diameter d on n vertices is not more than $(\frac{2^{\lceil \frac{d}{2} \rceil}-1}{\lceil \frac{d}{2} \rceil})n + O(1)$. Several results on $f(G)$ for graphs of diameter one, two, and three were previously known. If G is a graph of diameter 2, Patcher, Snevily, and Voxman [2] showed by case-checking

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that $f(G) \leq n+1$. If G is a graph of diameter 3, Boris Bukh [1] proved that $f(G) \leq \frac{3}{2}n + O(1)$ by examining the paths in the graph. Our approach is to partition the vertex set into pieces which can be examined individually.

2. Endpoints and Sections

Given any target t on a connected graph G, and given a maximal saturation distribution D, we define the following terms.

Definition 1. A vertex v is called an *endpoint* if $p(v) = 1$. Let $EP(G)$ denote the set of endpoints in G.

Definition 2. A *section* S of a graph G is a connected induced subgraph of $G \setminus t$ such that $N(S) \subseteq EP(G) \cup \{t\}$, where $N(S)$ denotes the neighbors of S, and for any vertex $v \notin V(S)$, v is not pebblable by S.

Note: $V(G) \setminus \{t\}$ is a section. Let $\bar{S}$ be the induced subgraph of G on the vertex set $V(S) \cup \{t\}$.

Definition 3. A *sectioning* of G is a collection of sections $S^* = \{S_i\}_{i=1}^{k}$ such that the following conditions are satisfied:

1. $V(S_i) \cap V(S_j) = \varnothing$ for $i \neq j$;
2. $\forall v \in V(G) \setminus \{t\}$, $v \in V(S_i)$ for some $1 \leq i \leq k$.

Let M be the maximum number of sections possible for any sectioning of G. Then if $|S^*| = M$ we call the sectioning S^* a *refined sectioning* of G.

Lemma 1. *For any connected graph G, there exists a unique refined sectioning of G.*

Proof. Let $V(G) = \{v_1, v_2, \cdots, v_{n-1}, t\}$ where $n = |V(G)|$

Let X be a collection of sets $\{x_i\}_{i=1}^{n-1}$ where $x_i = \{v_i\}$ for $1 \leq i \leq n-1$.

For each set $x_i \in X$, look at the vertices pebblable by x_i. For each vertex $w_i \notin x_i$ pebblable by x_i, since $w_i \neq t$, then $w_i \in y_i$ for some set $y_i \in X$.

x_i and y_i must be in the same section. Replace x_i and y_i in X by the set $z_i = x_i \cup y_i$.

We begin with a finite number of sets in X and everytime two sets are replaced by their union, $|X|$ decreases by 1. Therefore, after a finite number of iterations through the sets of X, we must reach a point where for every set $x_i \in X$, $\{$the set of vertices pebblable by $x_i\} = x_i$.

Since each set $x_i \in X$ is a section and each section is its minimum size, then X is the unique refined sectioning of G. $\square$

Definition 4. The *length* of a section S is $l(S) = max_{v \in V(S)}\{d(v,t)\}$.

Remark 1. It follows from the definition that $1 \leq l \leq d$.

Lemma 2. *Given a refined sectioning S^* of G, for any two sections of S^*, say S_1 and S_2, $l(S_1) + l(S_2) \leq d+1$.*

Proof. Suppose $l(S_1) + l(S_2) \geq d+2$. Then $\exists u \in V(S_1)$ and $v \in V(S_2)$ such that $d(u,t) = l(S_1)$ and $d(v,t) = l(S_2)$.

Since sections can only be connected by their endpoints, then $d(u,v) \geq (l(S_1) - 1) + 1 + (l(S_2) - 1) = d+1$, which is a contradiction. $\square$

Corollary 1. *There can be at most one section with length $> \lceil \frac{d}{2} \rceil$.*

Definition 5. Let the *ratio* of a section be $r(S) = \frac{\{\#\text{ pebbles on } V(S)\}}{|V(S)|}$.

Lemma 3. *$r(S)$ is maximal when S is a path of maximum length.*

Proof. Let H be the subgraph $\bar{S}$ with edges added so that $EP(S) \cup \{t\}$ forms a clique. Connecting any two endpoints does not lower the pebbling number, so $f(H,t) = f(S,t)$.

Since H is connected, there exists a shortest path spanning tree T on H with start vertex t.

By Fan Chung's paper [3], $f(T,t) = 2^{a_1}+2^{a_2}+\cdots+2^{a_t}-t+1$, where $a_1, a_2, \cdots, a_t$ corresponds to the lenghts of the paths in the maximum path partition of T. Since T is a subgraph of H, $f(T) \geq f(H)$. Hence, $r(T) \geq r(H) = r(S)$.

Since $2^a + 2^b \leq 2^{a+b}$ for $a, b \geq 1$, $f(T,t)$ is maximal when T is a tree with the maximum number of disjoint paths of length $l(S)$. Since the pebbling number of a path of length l is 2^l, then $r(S) \leq \frac{2^l-1}{l}$. □

3. An Upper Bound on $f(G)$ for a Graph of radius r

Theorem 1. *If G is a connected graph on n vertices, then $f(G) \leq (\frac{2^r-1}{r})n + O(1)$, where r is the radius of G.*

Proof. By definition, there exists a vertex $v \in V(G)$ such that $d(v,u) \leq r$ for all $u \in V(G)$. Let T denote a shortest path tree with v as the root. Let the path from v to any target t be $P = \{v, v_1, v_2, \cdots, t\}$.

A maximal saturation distribution on T is determined by finding a maximum path partition [3]. In $f(T,v)$, all paths have a length $\leq r$, and hence, have a ratio $\leq \frac{2^r-1}{r}$.

We will examine $f(T,v_i) - f(T,v_{i+1})$. When the target is moved from v_i to v_{i+1}. Only one path in the maximum path partition will be extended by a length one. All other paths either remain unchanged, or decrease in length. Hence, $f(T,v_i) - f(T,v_{i+1}) \leq 2^d$.

Since the target needs to be moved at most $\frac{d}{2}$ times, then $f(G,t) \leq f(T,t) \leq f(T,v) \leq (\frac{2^r-1}{r})n + O(1)$.

□

4. An Upper Bound on $f(G)$ for Graphs of Diameter d

Definition 6. A graph is a *good* graph if for any section S in the refined sectioning of G, at least one of the following is satisfied:

(1) $|EP(S)| \leq 2^d$;
(2) $r(S) \leq \frac{2^{\lceil \frac{d}{2} \rceil}-1}{\lceil \frac{d}{2} \rceil}$.

Theorem 2. *If G is a good, connected graph on n vertices, then $f(G) \leq (\frac{2^{\lceil \frac{d}{2} \rceil}-1}{\lceil \frac{d}{2} \rceil})n + O(1)$*

Proof. Let $S^{\max}$ be a section of maximum length in the refined sectioning of G. From [previous theorem], there can be at most one section with length $> \lceil \frac{d}{2} \rceil$.

We will induct on $l(S^{\max}) - \lceil \frac{d}{2} \rceil$.

When $l(S^{max}) - \lceil \frac{d}{2} \rceil \leq 0$, then every section has length $\leq \lceil \frac{d}{2} \rceil$. Then the maximum ratio for any section is $\leq \frac{2^{\lceil \frac{d}{2} \rceil}-1}{\lceil \frac{d}{2} \rceil}$ and $f(G)$ is as desired.

Assume $f(G) \leq (\frac{2^{\lceil \frac{d}{2} \rceil - 1}}{\lceil \frac{d}{2} \rceil})n + O(1)$ for $l(S^{\max}) - \lceil \frac{d}{2} \rceil = k \geq 0$.

When $l(S^{\max}) - \lceil \frac{d}{2} \rceil = k + 1$, let $T^{\max}$ be a shortest path tree on $\bar{S}^{\max}$ with t as the start vertex. If $r(S^{\max}) \leq \frac{2^{\lceil \frac{d}{2} \rceil - 1}}{\lceil \frac{d}{2} \rceil}$, we are done.

Since G is good, we may assume $EP(S^{\max}) \leq 2^d$. Therefore, in any path partition of $T^{\max}$, there can be at most 2^d paths of length $l(S^{\max})$.

Let the new target set be $t^* = \{t\} \cup EP(S^{\max})$. Then the length of the maximum section with this new target is $l(S^{\max}) - 1 = k$.

Then by the induction hypothesis $f(G, t^*) \leq (\frac{2^{\lceil \frac{d}{2} \rceil - 1}}{\lceil \frac{d}{2} \rceil})n + O(1)$

Then when the target is reduced to the original vertex {t}, looking at $S^{\max}$, at most 2^d paths are extended by length 1. Thus, the maximum increase in $f(G, t)$ from $(f(G, t^*)$ is $\leq 2^d * 2^d$. Since the target can only be "extended" at most $\lfloor \frac{d}{2} \rfloor$ and thus $f(G, t) \leq (\frac{2^{\lceil \frac{d}{2} \rceil - 1}}{\lceil \frac{d}{2} \rceil})n + O(1)$ as desired. ☐

5. Additional Problems

The main question that still remains to be solved is whether all graphs are good or not. Experimentally, we have not found any graphs that violate the good graph property. In fact, condition (1) in the good graph property could be changed to $|EP(S)| \leq g(d)$ for at least one funtion g of d and the main result would still hold.

The implied constant is very large, and could be immediately improved by a more careful investigation of how the pebbling number changes when moving the target.

An (r, m) *pebbling move* consists of removing r pebbles from one vertex and then placing m pebbles at an adjacent vertex.

In this section, we describe an algorithm for finding $f_{r,m}(G)$ explicitly for paths as well as the pebbling number of even cycles. We will also discuss the pebbling number or odd cycles for when $r|m$ and a possible upper bound for when $r \nmid m$.

6. (r, m) Pebbling of Paths

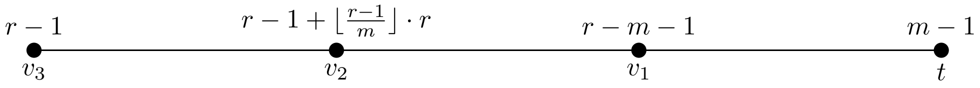
The (2,1) pebbling number of a path is known to be 2^n where n is the length of the path. We will discuss an algorithm for finding the (r, m) pebbling number of paths.

Algorithm for finding the (r, m) pebbling number of paths

Let $v_n v_{n-1} v_{n-2} ... v_2 v_1 t$ be a path of length n with t as the target vertex. Begin by saturating the path with $m - 1$ pebbles on the target vertex and $r - 1$ pebbles an every other vertex.

Next, find the number of pebbles were needed to achieve as many of the $r - 1$ pebbles that are on $v - 1$ as possible, and add that to the $r - 1$ pebbles on v_2. Intuitively, that number is $\lfloor \frac{r-1}{m} \rfloor \cdot r$.

Now the pebbling of the graph looks like:



Repeating this step by moving pebbles from v_2 to v_3, the saturated graph will contain $r-1+\lfloor\frac{r-1}{m}\rfloor\cdot r$-$\lfloor\frac{r-1+\lfloor\frac{r-1}{m}\rfloor\cdot r}{m}\rfloor\cdot m$ pebbles on v_2 and $r-1+r-1+\lfloor\frac{r-1}{m}\rfloor\cdot r$-$\lfloor\frac{r-1+\lfloor\frac{r-1}{m}\rfloor\cdot r}{m}\rfloor\cdot r$ pebbles on v_3.

We introduce notation to represent the number of pebbles on each vertex in order to present this algorithm as a recursive formula.

For P_n, a path of length n, let v_j be the vertex that is a distnance j away from the target, $v_j \in V(P_n)$ and $1 \le j \le n$. Also let $\hat{f}(v_j)$ represent the maximum number of pebbles on vertex v_j in which the saturated state of the path is maintained. Let $\hat{g}(v_i)$ be the number of pebbles on v_i $(1 \le i < j)$ that maintains the saturated state of the path when $\hat{f}(v_j)$ pebbles are placed on v_j.

It is impossible to move any pebbles from t to v_1, so we begin with the pebbles on v_1. Begin the algorithm at $\hat{f}(v_1) = r-1$ and do an iteration for $j = 1, 2, 3, ..., n$.

$$\hat{f}(v_j) = \lfloor\tfrac{\hat{f}(v_{n-1})}{m}\rfloor\cdot r + (r-1)$$

$$\hat{g}(v_i) = \hat{f}(v_i) \bmod m \qquad \text{where } 1 \le i < j$$

$$\hat{g}(v_k) = r-1 \qquad \text{where } j < k \le n$$

By use of this algorithm, the pebbling number of a path of length n would be

$$f_{r,m}(P_n) = \hat{f}(v_n) + \textstyle\sum_{i=1}^{n-1}\hat{g}(v_i) + (m-1) + 1$$

When $m|r$, $\hat{f}(v_n) = m(\frac{r}{m})^k$. Notice that with each iteration, a number of pebbles remains on each vertex (denoted by $\hat{g}(v_i)$). When $m|r$, $(m-1)$ pebbles are left on each intermediate vertex,therefore the pebbling number of a path of length n when $m|r$ is

$$f_{r,m}(P_n) = m(\tfrac{r}{m})^n + (m-1)(n-1) \qquad \text{when } m|r$$

However, when $m \nmid r$, there exists a more complicated pattern to a path's $\hat{g}(v_i)$. For the same path $v_n v_{n-1} v_{n-2} ... v_2 v_1 t$ of length n, it is clear that $\hat{g}(t) = m-1$ and $\hat{g}(v_1) = (r-1) \bmod m$. To find $\hat{g}(v_2)$ without having to compute $\hat{f}(v_2)$

$$\hat{g}(v_2) = [\hat{g}(v_1) + (r-m)] \bmod m.$$

Continue the pattern

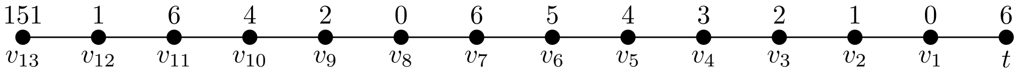
$$\hat{g}(v_{i-1}) + (r-m) \bmod m \qquad 1 \le i < n$$

until the shortest sequence $\hat{g}(v_1), ..., \hat{g}(v_i)$ ceases to be a sequence of increasing numbers. At this point, we start at $\hat{g}(v_i)$ to find $\hat{g}(v_j)$.

$$\hat{g}(v_{i+j}) = \hat{g}(v_{i+j-1}) + 2(r-m) \qquad \text{for } 0 < j < n-i$$

We do this until the sequence $\hat{g}(v_i), \hat{g}(v_{i+1}), ..., \hat{g}(v_{j-1}), \hat{g}(v_j)$ is no longer increasing. We then add $3(r-m), 4(r-m), ...$ until we reach $\hat{g}(v_{n-1})$.

Example:
(8,7) Pebbling of a path of length 13



7. (r, m) Pebbling of Cycles

Theorem 3. *(Pachter, Snevily, Voxman)* [2] $f(C_{2k}) = 2^k$

$(2, 1)$ pebbling of a cycle of length $2k$ is the same as pebbling a path with a length equal to diameter of the cycle. We find the (r, m) pebbling number for even cycles of length $2k$ for which we prove necessity, and sufficiency can be proved by case-checking.

Theorem 4. $f_{r,m}(C_{2k}) = m \cdot (\frac{r}{m})^k \ + (m-1) \cdot (n-1)$

Proof. Let $C_{2k+1} = xv_{k-1}v_{k-2}...v_2v_1tu_1u_2...u_{k-2}u_{k-1}x$ and let t be our target vertex.Let P_V be that path $xv_{k-1}v_{k-2}...v_2v_1t$ and P_U be the path $u_{k-1}u_{k-2}...u_2u_1t$. Note that $f_{r,m}(P_V) = f_{r,m}(P_U) = m(\frac{r}{m})^{k-1}$.

First we prove necessity. Suppose we are given $m \cdot (\frac{r}{m})^k - 1 + (m-1)(2k-2)$ pebbles. Place $m \cdot (\frac{r}{m})^k - 1$ pebbles on x and $(m-1)$ pebbles on each remaining vertex of C_{2k}. $m \cdot (\frac{r}{m})^{k-1} - 1$ pebbles is the potential of v_{k-1} (or u_{k-1}). That is clearly not enough to pebble the entire cycle.

□

Theorem 5. *(Pachter, Snevily, Voxman)*[2] $f(C_{2k+1}) = 2\lfloor \frac{2^{k+1}}{3} \rfloor$ *when* $m|r$

Pachter, Snevily and Voxman proved the (2,1) pebbling number of an odd cycle of length $2k+1$. We have the (r, m) pebbling number of a cycle of length $2k+1$ for which we prove necessity, and sufficiency can proved by case-checking.

Theorem 6. $f_{r,m}(C_{2k+1}) = 2\cdot\ [m\cdot\ (\frac{r}{m})^k - 1 - \lfloor\ (\frac{r}{m})^{k-1} \cdot (\frac{r}{r+m})\rfloor \cdot m] + (m-1)(2k-1) + 1$

Proof. Let $C_{2k+1} = v_kv_{k-1}v_{k-2}...v_2v_1tu_1u_2...u_{k-2}u_{k-1}u_kv_k$ be the odd cycle of length $2k+1$ and let t be our target vertex. Let P_V be the path $tv_1v_2...v_{k-2}v_{k-1}v_k$ of length k, and let P_U be the path $tu_1u_2...u_{k-2}u_{k-1}u_k$ also of length k. Note that $f_{r,m}(P_V) = f_{r,m}(P_U) = m \cdot (\frac{r}{m})^k + (m-1) \cdot (n-1)$. First, we show necessity.

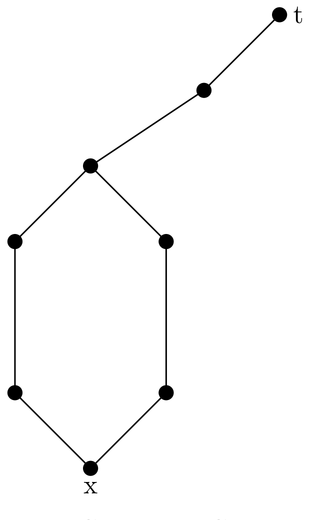
Suppose we are given $2\cdot$ $[m\cdot (\frac{r}{m})^k - 1 - \lfloor (\frac{r}{m})^{k-1} \cdot (\frac{r}{r+m}) \rfloor \cdot m] + (m-1)(2k-1)+1$ pebbles. Place $[m\cdot (\frac{r}{m})^k - 1 - \lfloor (\frac{r}{m})^{k-1} \cdot (\frac{r}{r+m}) \rfloor \cdot m]$ pebbles on v_k and $[m\cdot (\frac{r}{m})^k - 1 - \lfloor (\frac{r}{m})^{k-1} \cdot (\frac{r}{r+m}) \rfloor \cdot m]$ pebbles on u_k and $(m-1)$ pebbles on each of the remaining vertices of C_{2k+1}. Suppose we make an (r, m) pebbling move from u_k to v_k. Then we will have $m(\frac{r}{m}) - 1$ pebbles at v_k. Since $m|r$, another pebbling move from v_k to v_{k-1} places $m(\frac{r}{m})^{k-1} + (m-1)(k-1)$ pebbles on P_v. This will not pebble t.

□

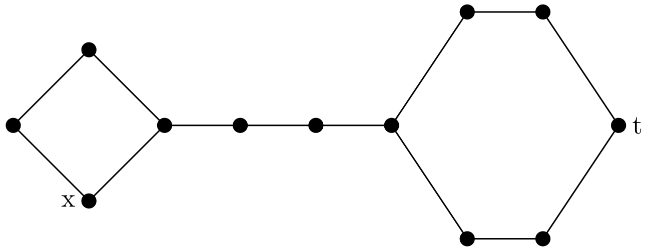
8. The Two Pebbling Property

We define q to be the number of vertices containing pebbles and then we say a graph satisifies the two pebbling property, if given $2f(G) - q + 1$ pebbles in any configuration, two pebbles can be moved to any specified target vertex. It is known that all trees, cycles, and diameter two graphs have the two pebbling property. Because all trees and cycles have it, it was conjectured by Snevily that all bipartite graphs also have the two pebbling property S. This paper proves the result for two subclasses of bipartite graphs, $C_{2n} * P_m$ and $C_{2m} * P_k * C_{2n}$, shown below.

$C_{2n} * P_m$:



$C_{2m} * P_k * C_{2n}$:



Lemma 4. *The pebbling number of* $C_{2n} * P_m$ *is* 2^{n+m}.

Proof. Let$G = C_{2n} * P_m$ with vertex set $tv_1v_2 \dots v_{m-1}ya_1a_2 \dots a_{n-1}xb_{n-1} \dots b_2b_1$ such that x and t are the ends of the two longest paths. Call $xv_1v_2 \dots v_{m-1}y, P_m$, $a_1a_2 \dots a_{n-1}, P_a$, and $b_{n-1} \dots b_2b_1, P_b$. Without loss of generality is enough to consider the longest path. Let t be the target vertex in our graph. Note that there are two ways to get from x to t: $P_a \cup P_m and P_b \cup P_m$.Because the the pebbling number of a graph is at least 2^d, $f(C_{2n} * P_m) \geq 2^{n+m}$. So it is enough to show that 2^{n+m} will pebble the graph. Assume there are no pebbles on x. Then the 2^{m+n} pebbles are divided amongst two paths, both of length $m + n - 1$. One path is sure to have at least 2^{m+n-1} pebbles on it, allowing a pebble to reach t.
Now assume there are pebbles on x. If either $P_a \cup P_m$ or $P_b \cup P_m$ have 2^{m+n-1} pebbles, then a pebble can reach t and we are finished, so assume neither one does. In fact, let the number of pebbles on $P_a \cup P_m$ be $2^{m+n-1} - s$ and on $P_b \cup P_m$ let there be $2^{m+n-1} - r$ pebbles. Then x must have at least $s + r$ pebbles. If $s = r$, then without loss of generality, r pebbles can be moved to $P_b \cup P_m$ and t can be pebbled. Now suppose $s > r$ then x has greater than $2r$ pebbles on it, and r pebbles can be moved to $P_b \cup P_m$, allowing a pebble to reach t. Thus, $f(G) = 2^{m+n}$. □

Definition 7. μ_{vertex} is the number of pebbles on a given vertex.

Definition 8. Given a vertex, d_i, on a path, the weight, w_{d_i} of that vertex is $2^i * \mu_{d_i}$. For a path of length n,

$$\sum_{i=0}^{n} 2^i * \mu_{d_i}$$

is the weight of the path, w_p.

Lemma 5. *SIn a path of length n, if* $w_p \geq k * 2^n$ *then at least k pebbles can reach the target vertex.*

The proof of the following theorem follows closely to that of Snevily and Foster's that all cycles have the 2-pebbling property S.

Theorem 7. *All lasso graphs have the two pebbling property.*

Proof. Let $G = C_{2n}*P_m$ with vertex set $\{t, v_1, v_2 \dots v_{m-1}, y, a_1, a_2 \dots a_{n-1}, x, b_{n-1} \dots b_2, b_1\}$ such that x and t are the ends of the two longest paths. Call $xv_1v_2 \dots v_{m-1}y, P_m$, $a_1a_2 \dots a_{n-1}, P_a$, and $b_{n-1} \dots b_2b_1, P_b$. Without loss of generality, it is enough to consider the longest path. Let t be the target vertex in our graph. Notice that $q \prec 2^{m+n-1}$. Now, assume $\mu_x < 2^{m+n} - q + 2$. Then $2 * 2^{m+n} - q + 1 - \mu_x \geq 2^{m+n}$. Once again, either $P_m \cup P_a$ or $P_m \cup P_b$ must have 2^{m+n-1} pebbles, allowing a pebble to reach t. Using at most 2^{m+n-1} pebbles leaves $2*2^{m+n} - q + 1 - 2^{m+n-1} \geq 2^{m+n}+1$

pebbles, allowing a second pebble to reach t.
Now, assume $\mu_x \geq 2^{m+n} - q + 2$. Let $q = 2r$ if q even, and $2r + 1$ if q odd. Then at least one of the paths must have r vertices with pebbles on them. Assume r is at least 3, now note that $2 + 2^2 + \ldots 2^r \geq 2q$. Then, using a pebble from each of the $a_j, 1 \leq j \leq r$ vertices on the path and $2^{m+n} - 2q$ pebbles from x, by lemma 2, a pebble can reach t. The cost of this is at most $2^{m+n} - 2q + r$ pebbles, leaving $2^{m+n} + 2q - r + 1$ pebbles, enough to pebble t again.

The cases when $q = 2, 3, 4, 5$ remain to be considered.

q=2: The two vertices must lie on the same path, and so by lemma 2, two pebbles will be able to reach *t*.

q=3: If all the vertices lie on the same path, then two pebbles can certainly reach *t*. So, assume they don't. So two of the vertices must lie on opposite sides of the cycle, say on b_i and a_j. Now assume that $\mu_{a_j} = \mu_{b_i} = 1$. Then $\mu_x = 2 * 2^{m+n} - 4$ Because the three vertices do not lie on the same path, both i and j must be at least 2. Therefore, by lemma 2, it is possible to get 2 pebbles to *t*. So, without loss of generality, assume $\mu_{a_j} \geq 2$. Then, $2^{m+n-1} - 2$ pebbles can be moved to a_j at the cost of $2^{m+n} - 4 + 2 = 2^{m+n} - 2$ pebbles. This leaves $2 * 2^{m+n} - 2 - (2^{m+n} - 2) = 2^{m+n}$ pebbles, enough to send a second pebble to *t*.
The proof when q=4,5 is similar and shall be omitted. ☐

Lemma 6. *The pebbling number of* $C_{2m} * P_k * C_{2n}$ *is* 2^{m+k+n}.

Proof. Let $G = C_{2m} * P_k * C_{2n}$ with vertex set $\{u, a_1, \ldots a_{m-1}, x, b_{m-1} \ldots b_2, b_1, g_1, \ldots g_{k-1}, w, h_1, h_2 \ldots h_{n-1}, x, v_{n-1}, \ldots v_2, v_1\}$ such that *u* and *x* are the endpoints of the longest paths, of which there are four. Consider the endpoints, and let *x* be the target. Assume that $\mu_u = 0$. Then one of the longest paths must have at least $2^{m+k+n-2}$ pebbles. Then the weight of the path must be at least $2 * 2^{m+k+n-2}$, allowing a pebble to reach *x*. Now, assume $\mu_u > 0$. If any of the paths have $2^{m+k+n-1}$ pebbles, then it is easy to see that a pebble can reach *x*, so assume that none of them do. Then label $\mu_{eachpath} = 2^{m+k+n} - r$, $2^{m+k+n} - s$, $2^{m+k+n} - h$, or $2^{m+k+n} - k$ pebbles. Then *u* has at least $r + s + h + k$ pebbles, and $\lfloor \frac{r+s+h+k}{2} \rfloor$ can reach any one of the four paths. Certainly $\lfloor !cr + s + h + k2 \rfloor$ is at least *r*, *s*, *h*, or *k*, and so for one of the paths it is possible to get $2^{m+k+n-1}$ pebbles, and thus get a pebble to *x*. ☐

Theorem 8. *All graphs of the form* $C_{2m} * P_k * C_{2n}$ *have the 2-pebbling property.*

Proof. Assume $\mu_u < 2^{m+k+n} - q + 2$. Then $2 * 2^{m+k+n} - q + 1 - \mu_u \geq 2^{m+k+n}$. The first pebble can thus reach *x* at a cost of at most $2^{m+k+n-1}$. This leaves $2 * 2^{m+k+n} - q + 1 - 2^{m+k+n-1} \geq 2^{m+k+n} + 1$ pebbles, allowing another pebble to reach *x*. Now, consider when $\mu_u \geq 2^{m+k+n} - q + 2$. Some path in G must have at least $\frac{q}{4}$ vertices with pebbles. Assume $q \geq 17$ note that this means $2 + 2^2 + \ldots + 2^{\frac{q}{4}} \geq 2q$. And so, using one pebble from each of the vertices and $2^{m+k+n} - 2q$ pebbles from *u*, it is possible, by lemma 2, to get a pebble to *x* at a cost of $2^{m+k+n} - 2q + \frac{q}{4}$ pebbles, leaving $2^{m+k+n} + q - \frac{q}{4} + 1$ pebbles, surely enough for another pebble to reach *x*.

The cases where $q < 17$ remain to be examined.

q=2: Then, as one of the vertices must remain u, the vertices must lie on the same path, and so by *lemma 2*, two pebbles will certainly be able to reach x.

q=3: If the vertices lie on the same longest path, then by lemma 2, we are finished, so assume they do not lie on the same path. Then they lie on opposite sides of the same cycle. Assume they lie on a_i and b_j, and further, assume $\mu_{a_i} = \mu_{b_j} = 1$ Then $\mu_u = 2 * 2^{m+k+n} - 4$. At the cost of $\frac{2*2^{m+k+n}-4}{2}$, $2^{m+k+n-1} - 1$ pebbles can be moved to each of the paths. Each of the paths now have a weight of at least $2^{m+k+n-1}$, allowing a pebble from each path to be moved to x. Now assume, without loss of generality, that $\mu_{a_i} \geq 2$ Then by moving $2^{m+k+n-1} - 2$ pebbles to a_i, at a cost of $2^{m+k+n} - 2$ pebbles, it is possible to get a pebble to x. This leaves 2^{m+k+n} pebbles, enough to send another pebble to x.
The cases for $4 \leq q \leq 16$ are similar and will be omitted. ☐

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