# The Irredundant Number of Star Graphs

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**Abstract**

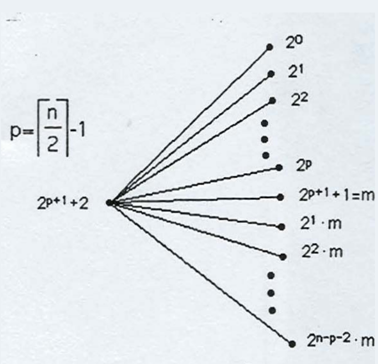
An irredundant number of a graph is a vertex labeling problem on simple graphs. In such a labeling, it is possible for two distinct connected induced subgraphs $H_1$ and $H_2$ to have the same label sum. If a labeling has the property that $\sum_{x \in H_1} l(x) = \sum_{x \in H_2} l(x) \iff H_1 = H_2$ it is called *irredundant* and the smallest label sum of an irredundant labeling is called the *irredundant number* of $G$, denoted $\lambda(G)$.

## 1 Introduction

Let G be a $K_{1,n}$ graph with a *vertex labeling* being a mapping from V(G) to $Z^+$. A labeling $l$ of $G$ is called *irredundant* if, for all connected induced subgraphs $H_1$ and $H_2$, $\sum_{x \in V(H_1)} l(x) = \sum_{x \in V(H_2)} l(x)$ if and only if $H_1 = H_2$ [1]. If the label sum is the smallest sum possible, the label sum is the *irredundant number* of G. In this report, we show a good upper bound for the irredundant number and prove that it is in fact irredundant.

## 2 An Irredundant Labeling of a $K_{1,n}$ graph

**Theorem 1.** *Let $p+1$ vertices of a $K_{1,n}$ graph be labeled $2^0, 2^1, 2^2, \dots, 2^p$ where $p = \lceil \frac{n}{2} \rceil - 1$. Let the next leaf be labeled $2^{p+1} + 1 = m$, and the center of the star be labeled $m + 1$. Let the remaining $n-p-2$ leaves be labeled $2m, 2^2m, \dots, 2^{n-p-2}m$. If a $K_{1,n}$ graph is labeled this way, than the label sum the graph provides a good upper bound for $\lambda(K_{1,n})$.*




**Comment 1.** *The label sum of $K_{1,n}$ is $2^n + 2^{p+1} + 2^{n-p-1}$, shown below.*

$$
\begin{aligned}
2^{p+1} - 1 + m(2^{n-p-1} - 1) + m + 1 &= 2^{p+1} + m \cdot 2^{n-p-1} && (1)\\
&= 2^{p+1} + (2^{p+1} + 1)(2^{n-p-1}) && (2)\\
&= 2^{p+1} + 2^{n} + 2^{n-p-1} && (3)
\end{aligned}
$$

To show that the above labeling is irredundant we consider different cases of two distinct connected induced subgraphs $H_1$ and $H_2$ of $K_{1,n}$ and prove $\Sigma_{x \in H_1} l(x) \neq \Sigma_{x \in H_2} l(x)$.

## 2.1 Part 1

If $H_1$ and $H_2$ are two different single vertices, clearly the labels on them are different.

## 2.2 Part 2

Suppose $H_1 = \{x\}$ and $H_2$ is a connected induced subgraph.

### Case 1

If $x = v$ then $l(v)$ must occur in the sum of labels of vertices in $H_2$. Hence $H_1$ and $H_2$ cannot have same label sum.

### Case 2

Assume $x = x_i$ for $1 \leq i \leq p+2$. Since $v \in V(H_2)$ and $l(v) > l(x_i)$ for $1 \leq i \leq p+2$, $H_1$ and $H_2$ cannot have the same label sum.

### Case 3

Assume $x = x_i$, $p+2 < i \leq n$. If $x_j \in V(H_2)$ for $j > i$ then $l(x_j) > l(x_i)$ and hence $H_1$ and $H_2$ cannot have the same label sum.

## 2.3 Part 3

Suppose $x_j \in V(H_2)$ are such that $j < i$.

### Case 1

If $x_j \in V(H_2)$ for $1 \leq j \leq i-1$, then

$$
\begin{aligned}
\Sigma_{j=1}^{i-2} l(x_j) + l(v) &= m + 1 + \sum_{j=0}^{p} 2^j + \sum_{j=0}^{i-p-3} m \cdot 2^j && (4)\\
&= m + 1 + 2^{p+1} - 1 + m(2^{i-p-2} - 1) && (5)\\
&= 2^{p+1} + m \cdot 2^{i-p-2} && (6)\\
&= 2^{p+1} + (2^{p+1} + 1) \cdot 2^{i-p-2} && (7)\\
&= 2^{p+1} + 2^{i-1} + 2^{i-p-2} && (8)\\
&= 2^{p+1} + l(x_i) && (9)\\
&> l(x_i) && (10)\\
& && (11)
\end{aligned}
$$

$\therefore$ the label sum of $H_2$ is greater than the label sum of$H_1$.

Further, in the instance that $\sum_{j=1}^{i-2}(x_j) + l(v) + l(x_q)$ where $i < q < n$, then the label sum of $H_2$ is still greater than $H_1$.

## Case 2

**Lemma 1.** *If $p+2 < i \leq n$, then $l(v) + \sum_{k=1}^{i-2} l(x_k) < l(x_i)$.*

*Proof.* Suppose $i = p+3$. Then

$$\begin{aligned}
l(v) + \sum_{k=1}^{i-2} l(x_k) &= m + 1 + \sum_{j=0}^{p} 2^j && (12)\\
&= m + 1 + 2^{p+1} - 1 && (13)\\
&= m + 1 + (m-1) - 1 && (14)\\
&= 2 \cdot m - 1 && (15)\\
&< 2 \cdot m && (16)\\
&= l(x_{p+3}) && (17)\\
& && (18)
\end{aligned}$$

$\therefore$ the label sum of $H_2$ is less than the label sum of $H_1$. □

If $p+3 < i \leq n$,

$$\begin{aligned}
l(v) + \sum_{k=1}^{i-2} l(x_k) &= l(v) + \sum^{p+1} l(x_k) + \sum^{i-2} l(x_k) && (19)\\
&= 2 \cdot m - 1 + m \sum_{j=o}^{i-p-4} 2^j && (20)\\
&= 2 \cdot m - 1 + m(2^{i-p-3} - 1) && (21)\\
&= m - 1 + 2^{i-p-3} m && (22)\\
&< 2^{i-p-3} m + 2^{i-p-3} m && (23)\\
&= 2^{i-p-2} m && (24)\\
&= l(x_i) && (25)\\
& && (26)
\end{aligned}$$

$\therefore$ The label sum of $H_2$ is less than the label sum of $H_1$.

**Lemma 2.** *If $p+2 < i \leq n$ and $x_{i-1} \in V(H_2)$ then $\sum_{x \in V(H_2)} l(x) \neq l(x_i)$.*

*Proof.* If $x_j \in V(H_2)$ for $1 \leq j \leq i-1$, then

$$\begin{aligned}
\sum_{x \in V(H_2)} l(x) &= l(v) + \sum_{j=1}^{i-1} l(x_j) && (27)\\
&= m + 1 + \sum_{j=0}^{p} 2^j + \sum_{j=0}^{i-p-3} 2^j \cdot m && (28)\\
&= 2 \cdot m - 1 + m(2^{i-p-2} - 1) && (29)\\
&> 2^{i-p-2} m && (30)\\
&= l(x_i) && (31)\\
& && (32)
\end{aligned}$$

$\therefore$ the label sum of $H_2$ is greater than the label sum of $H_1$. □

Case 4

If $x_j \notin V(H_2)$ for $p+2 \leq j \leq i-1$, then

$$\sum_{x \in V(H_2} l(x) < 2^{p+1} + l(x_i) - 2^{j-p-2}m = 2^{p+1} + l(x_i) - 2^{j-p-2}(2^{p+1}+1) \quad (33)$$

$$= 2^{p+1} + l(x_i) - 2^{j-1} - 2^{j-p-2} \quad (34)$$

$$\leq l(x_i) - 2^{j-p-2} \quad (35)$$

$$< l(x_i) \quad (36)$$

$$(37)$$

$\therefore$ the label sum of $H_2$ is less than the label sum of $H_1$

## 2.4 Part 4

*Proof.* The lemmas and cases above show that no single vertex will have the same subgraph label as any other subgraph because it will be either too large or too small. To prove the case where neither $H_1$ nor $H_2$ are single vertices, consider the following. Let the largest label be $l(x_i) \in V(H_2)$, and let all labels that are in both the label sum of $H_1$ and the label sum of $H_2$ be disregarded because they will cancel. Also, note that $l(v)$ will always cancel becase all single vertex cases have been considered, so each subgraph would contain l(v). Further, we know that $\sum_{x_j \in V(H_1)} l(x_j)$ for $1 \leq j \leq i-1$ is less than $l(x_i)$. Clearly, any label added to $H_2$ would only make the sum larger, and any label missing from the label sum of $H_1$ would only produce a smaller sum. $\therefore \sum_{x \in H_1} l(x) = \sum_{x \in H_2} l(x) \iff H_1 = H_2$.

□

# 3 A Conjecture on the Irredundant Number of Star Graphs

**Conjecture:** *Let* $p+1$ vertices of a $K_{1,n}$ graph be labeled $2^0, 2^1, 2^2, \ldots, 2^p$ where $p = \lceil \frac{n}{2} \rceil - 1$. Let the next leaf be labeled $2^{p+1}+1 = m$, and the center of the star be labeled $m+1$. Let the remaining $n-p-2$ leaves be labeled $2m, 2^2m, \ldots, 2^{n-p-2}m$. If a $K_{1,n}$ graph is labeled in such a way, then the resulting label sum is the irredundant number of the graph.

*Proof.* To follow . . . . □

### *Acknowledgements*

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# References

[1] Stephen Penrice, *Some New Graph Labeling Problems: A Preliminary Report*, S.U.N.Y. College at Portland, Portland, NY, 1995.