The Sum Number of a Unicyclic Graph

Jeff Ginn

Sivaram Narayan

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Abstract

If a labeling exists of a graph G such that vertices $x, y \in G$ labeled with distinct positive integers $L(x)$ and $L(y)$ share an edge iff the sum $L(x) + L(y) = L(Z)$ is the label of a vertex $z \in G$ then G is said to be a sum graph. The sum number of G, denoted $s(G)$, is the number of isolated vertices which must be added to G to make G a sum graph. We will give labelling schemes for the sum number of a cycle, unicyclic graphs, and a class of multicyclic graphs. Results on the sum number of unicyclic graphs and multicyclic graphs will be presented.

1 Introduction

The concept of the sum graph was first introduced by Harary [3]. A sum graph can be viewed as a representation of a compressed data structure. Compressing data is important in many applications involving computer memory and storage methods. For more information on and examples of the applications of sum graphs see [2], [5], or [6].

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2 Basic Definitions

Definition. If a labeling exists of a graph G such that vertices $x, y \in G$ labeled with distinct positive integers $L(x)$ and $L(y)$ share an edge iff the sum $L(x) + L(y) = L(Z)$ is the label of a vertex $z \in G$ than G is said to be a **sum graph**.

All sum graphs will have at least one isolated vertex.

Definition. The **sum number** of G, denoted $s(G)$, is the number of isolated vertices which must be added to G to make G a sum graph.

In [1] Ellignham proved that the sum number of a tree is one. In [6] the sum number of a complete graph K_n with $n \geq 4$ vertices is given as $s(K_n) = 2n - 3$.

3 Cycles

3.1 The Sum Number of Cycles

The sum number of cycles is found in [4]. For a cycle C_n containing n vertices:

$$s(C_4) = 3$$
$$s(C_n) = 2 \text{ for } n \neq 4$$

3.2 Labelling Cycles

A cycle with three vertices can be labeled $\{1, 2, 3\}$. The two isolated vertices will be labeled 5 and 4. A four cycle with vertex set $\{x_1, x_2, x_3, x_4\}$ and edge set $\{x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$ can be labeled with:

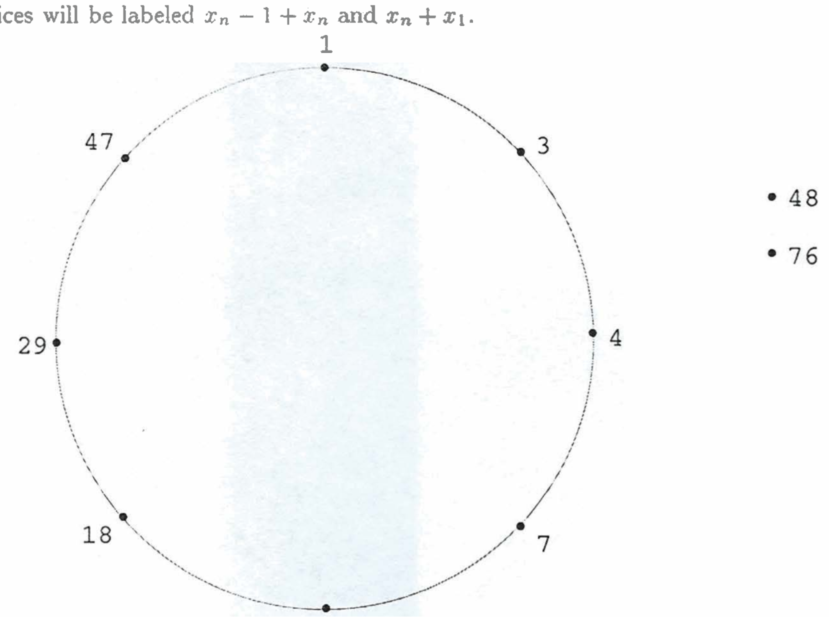
$$L(x_1) = 5$$
$$L(x_2) = 7$$
$$L(x_3) = 6$$
$$L(x_4) = 8$$

The three isolated vertices will be labeled 11, 12, and 13.

The result on $s(C_n)n \neq 4$ can be seen by taking the set of vertices of the cycle $\{x_1, x_2, ..., x_n\}$ with edge set $\{x_1x_2, x_2, x_3, ..., x_{n-1}x_n, x_nx_1\}$ and picking $L(x_1)$ and $L(x_2)$ to be any two distinct positive integers with the restriction that if $L(x_1) = 1$ then $L(x_2) \neq 2$. If $L(x_1) = 1$ and $L(x_2) = 2$ then the resulting labeling will contain an isolated vertex whose label is the sum of two other vertices of the graph which are not connected. Label the remaining vertices as follows:

$$L(x_3) = L(x_2) + L(x_1)$$
$$L(x_4) = L(x_3) + L(x_2)$$
$$L(x_i) = L(x_{i-1}) + L(x_{i-2})\ (3 \leq i \leq n).$$

The two isolated vertices will be labeled $x_n - 1 + x_n$ and $x_n + x_1$.



Example. In the above example 1 and 3 were chosen as the labels of vertices x_1 and x_2. The label of vertex x_3 is the sum $1+3 = 4$ and so on. The isolated vertices are the sums $48 = 47+1$ and $76 = 29+47$.

4 Unicyclic Graphs

4.1 The Sum Number of Unicyclic Graphs

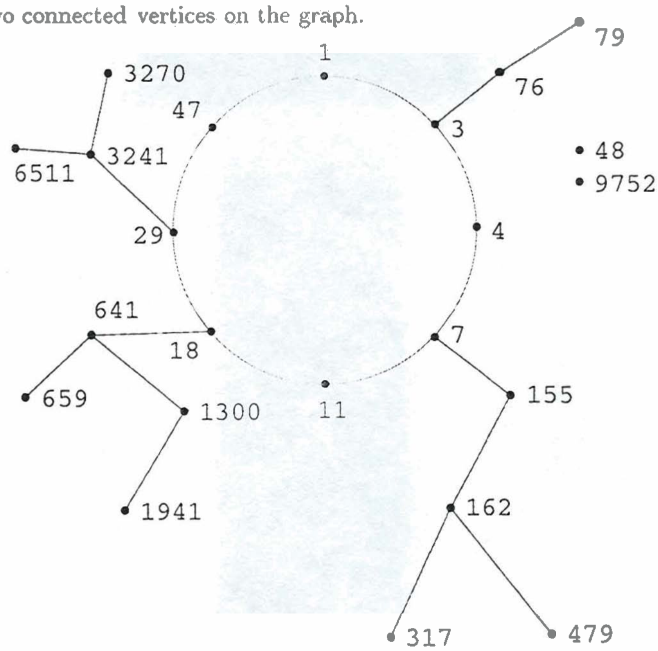
A graph which contains exactly one cycle is said to be a unicyclic graph, denoted U_{C_n}. Results on the sum number are:

$$s(U_{C_3}) = 1$$
$$s(U_{C_4}) = 3$$
$$s(U_{C_n}) = 3 \text{ for } n \geq 5$$

4.2 Labelling Unicycles

To label a unicyclic graph containing a cycle with three or four vertices proceed by labeling the cycle using the respective labeling for the cycle described above. Label the first vertex on a pendant which is connected to the three cycle 4. For a four cycle label two isolated vertices 11 and 12. Label the first vertex on a pendant which is connected to the four cycle 13. For each cycle, continue along the pendant labelling the next vertex as the sum of labels of the last two vertices labeled. If there is a branch, follow the longest path then return to the nearest branch and continue to label vertices along the branch as the sum of the labels of the last two labeled vertices. When a pendant is complete move to another pendant and label in the same fashion. After all pendants have been labeled one isolated vertex will be labeled with the largest label sum of connected vertices.

To label any U_{C_n} $n \neq 4$ so that $s(U_{C_n}) = 2$ begin by labelling the cycle using the method described above. Label one isolated vertex with the largest sum of the labels of two connected vertices on the cycle. Label a vertex adjacent to the cycle with the sum $x_n + x_1$. If there is another vertex on the pendant label it with the sum of the adjacent vertex and the vertex on the cycle. If the pendant branches follow any branch, labelling the next vertex on the path with the sum of the labels of the previous two vertices. When a path ends take the sum of the last two vertices labeled as the label of a vertex adjacent to the labeled path on the last branch or, if there are no more branches, as the label of a vertex adjacent to the cycle on another pendant. When all vertices on all pendants have been labeled, label the remaining isolated vertex as the sum of the labels of the last two vertices labeled. This should be the largest sum of the labels of any two connected vertices on the graph.



Example. The labelling of the unicycle above begins with the labelling of the cycle from the previous example. Instead of adding 76 as the label of an isolated vertex the first vertex in the pendant attached to the vertex of the cycle is labeled 76. The next vertex in the pendant is labeled with the sum $76+3 = 79$. The first vertex in the pendant attached to the vertex of the cycle labeled 7 is the sum $76 + 79 = 155$. Labelling of the vertices moves around the cycle in this manner. The second isolated vertex is labeled with the largest sum of labels of connected vertices $6511 + 3241 = 9752$.

5 Multicyclic Graphs

5.1 The Sum Number of Multcyclic Graphs

A graph which contains mutliple cycles is said to be a multicyclic graph. As a special class of multicyclic graphs we present a formula for the sum number of graphs which have at least one cycle and obey the property that each vertex is on no more than one cycle. The sum number of this class of multi-cyclic graphs can be generalized into two cases. If the graph contains one or more cycles with three vertices then:

$$N_3 + N_4 + N_{n \geq 5}$$

If the graph contains no cycles with three vertices then the sum number is:

$$N_4 + N_{n \geq 5} + 1$$

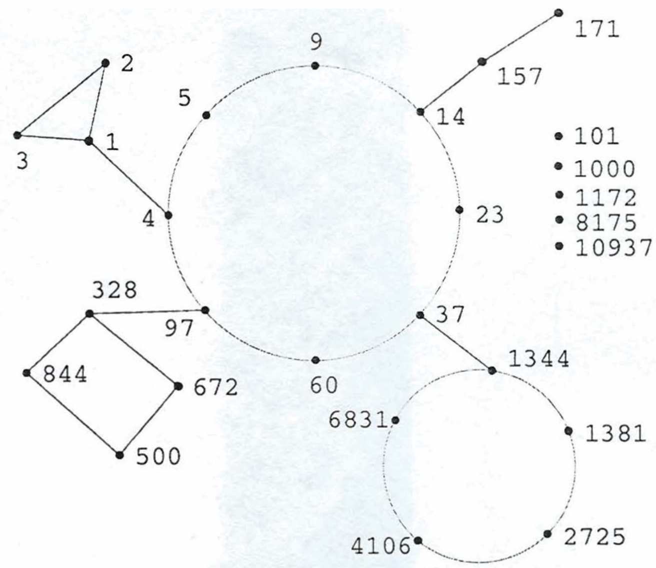
Here N_n represents the number of cycles in the graph with n vertices.

5.2 Labelling Multicycles

The labelling scheme for this type of graph follows from the labelling scheme for unicyclic graphs. If there is a cycle with three vertices then label it in the manner described above for labelling unicyclic graphs containing cycles with three vertices. If there is not a cycle with three vertices begin by labelling any cycle on the graph in the manner described above for unicyclic graphs with the appropriate number of vertices.

When a pendant leads to another cycle the first vertex on the cycle takes the label of the largest label sum of two connected vertices. If the cycle has four vertices, increment the label of this first vertex by one plus the largest label of a vertex on the graph which is not on the four cycle. Use this number as the label of the vertex opposite the first labeled vertex on the four cycle. Increment this label again by the same value the label of the first vertex was incremented and use this value as the label of another vertex on the cycle. Increment the last label one more time to get the label of the final vertex on the cycle. In this way, the sum of the labels of two of the connected vertices on the cycle is equal to the sum of the labels of the other two connected vertices on the cycle. For all other cycles the second vertex will be labeled with the sum of the label of the vertex on the pendant connected to the labeled vertex on the cycle. Labelling then proceeds recursively around the cycle.

Continue to label the graph in this fashion. Each cycle with four vertices will contribute two isolated vertices to the graph. All other cycles will contribute one isolated vertex to the sum number with the exception of the first three cycle which will not contribute any isolated vertices.



Example. In the above example of a multicycle the cycle with three vertices has been labeled in such a way as not to contribute any isolated vertices to the sum number of the graph. Sums of the labels of vertices coming from the three cycle led to the labels of the first vertices of the larger cycle. That cycle and the pendant edge coming off the vertex labeled 14 were labeled in the same manner the unicyclic graph in the above example was labeled. This cycle added the isolated vertex labeled 101 to the graph. The first vertex in the four cycle is labeled with the sum of the labels of the vertices of the pendant $157 + 171 = 328$. The opposite vertex has been labeled with 328 incremented by $171 + 1$ where 171 was the largest label not on the four cycle. The other two vertices of the cycle have been labeled by incrementing the largest label of a vertex on the cycle by 172. The four cycle adds isolated vertices with labels 1000 and 1172 to the graph. The remaining cycle is labeled in the usual manner for labelling cycles. This cycle adds the remaining two isolated vertices to the graph.

6 Methods of Proof

We are attempting to prove all conjectures listed above by contradiction. Allowing the sum number to be one less than it is conjectured to be in any of the above conjectures for any mapping of the sums of the labels of connected vertices onto the labels of the vertices of a graph results in a contradiction. The difficulty is in generalizing these contradictions into a proof.

Example. Consider the graph C_4 with two isolated vertices having vertex set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and edge set $\{x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$. We want to map the four sums of the labels of connected vertices:

$$L(x_1) + L(x_2)$$
$$L(x_2) + L(x_3)$$
$$L(x_3) + L(x_4)$$
$$L(x_4) + L(x_1)$$

onto the vertex set. By the definition of the sum number at least two vertices will be labeled with a label that is not the sum of labels of other vertices on the graph. We arbitrarily choose $L(x_1)$ and $L(x_2)$

for this mapping. For this mapping we map the sums to the labels of the remaining isolated vertices as follows:

$$
\begin{aligned}
L(x_1) + L(x_2) &\mapsto L(x_3) \\
L(x_2) + L(x_3) &\mapsto L(x_4) \\
L(x_3) + L(x_4) &\mapsto L(x_5) \\
L(x_4) + L(x_1) &\mapsto L(x_6)
\end{aligned}
$$

The contradiction here is $L(x_6) + L(x_2) = L(x_5)$ where $\{x_2x_6\}$ is not a member of the edge set. This is true because, by our mapping:

$$L(x_6) + L(x_2) = L(x_4) + L(x_1 + L(x_2) = L(x_4) + L(x_3) = L(x_5)$$

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