The Pebbling Number and The Optimal Pebbling Number of Unicyclic Graphs

Gregory T. Folsom
Central Michigan University
Research Experience for Undergraduates
Summer Research Scholars Program
Advisor: Dr. Sivaram Narayan

During the summer of 2002 at Central Michigan University under the guidance of Dr. Sivaram Narayan, I conducted research sponsored by the Summer Research Scholars Program. I researched a problem in the area of graph theory regarding the pebbling number and the optimal pebbling number of two classes of unicyclic graphs, in particular, "lassos" and "suns". In general, lassos can be defined as a cycle with n vertices adjoined to a path of length m and a sun can be defined as any number of paths of length one adjoined to the n vertices of a cycle. My research has culminated in the attached paper.

PROPOSITION 1

Let $n \in Z \geq 2$. Let $m \in Z \geq 1$. Then, $f(C_{2n} \cdot P_m) = 2^{n+m}$

Proof.

Let $G = C_{2n} \cdot P_m$.

$V(G) = \{V(C_{2n})\} \cup \{V(P_m)\} = \{v_0, v_1, v_2, \ldots, v_{2n-1}\} \cup \{v_0, v_{p_1}, v_{p_2}, \ldots, v_{p_{m-1}}, v_{p_m}\}$.

$E(G) = \{E(C_{2n})\} \cup \{E(P_m)\} = \{v_0v_1, v_1v_2, \ldots, v_{2n-1}v_0\} \cup \{v_0v_{p_1}, v_{p_1}v_{p_2}, \ldots, v_{p_{m-1}}v_{p_m}\}$

$diam(G) = d(v_n, v_{p_m}) = n + m$.

WLOG, we will designate v_{p_m} our target vertex.

Since $diam(G) = n + m$, $x = 2^{n+m} - 1$ is the maximum number of pebbles that can be placed at v_n such that a pebble could not be moved to v_{p_m}. Suppose x pebbles are placed at v_n. Let the path $P_A = \{v_n v_{n-1} \ldots v_1 v_0 \ldots v_{p_{m-1}} v_{p_m}\}$ and let the path $P_B = \{v_n v_{n+1} \ldots v_{2n-1} v_0 \ldots v_{p_{m-1}} v_{p_m}\}$. Since $l(P_A) = l(P_B) = diam(G)$, if a pebble were placed on a vertex in either P_A or P_B, then it would be possible, with the x pebbles at v_n, to move a pebble to v_{p_m} via the path in which the pebble was placed. Thus, no additional pebbles can be placed on the graph. Since the pebbling number is one more than the maximum number of pebbles that can be placed on the graph such that a pebble could not be moved to every vertex in the graph, the pebbling number is $x + 1$. Therefore, $f(C_{2n} \cdot P_m) = 2^{n+m}$. ☐

PROPOSITION 2

Let $n, m \in Z \geq 1$. Then, $f(C_{4n+1} \cdot P_m) = 2^{2n+m} + 2^n - 1$.

Proof.

Let $G = C_{4n+1} \cdot P_m$.

$V(G) = \{V(C_{4n+1})\} \cup \{V(P_m)\} = \{v_0, v_1, v_2, \ldots, v_{4n}\} \cup \{v_0, v_{p_1}, v_{p_2}, \ldots, v_{p_{m-1}}, v_{p_m}\}$.

$E(G) = \{E(C_{4n+1})\} \cup \{E(P_m)\} = \{v_0v_1, v_1v_2, \ldots, v_{4n}v_0\} \cup \{v_0v_{p_1}, v_{p_1}v_{p_2}, \ldots, v_{p_{m-1}}v_{p_m}\}$.

$diam(G) = d(v_{2n}, v_{p_m}) = 2n + m$.

WLOG, we will designate v_{p_m} our target vertex.

Since $diam(G) = 2n + m$, $x = 2^{2n+m} - 1$ is the maximum number of pebbles that can be placed at v_{2n} such that a pebble could not be moved to v_{p_m}. Suppose x pebbles are placed at v_{2n}. Let the path $P_A = \{v_{2n} v_{2n-1} \ldots v_1 v_0 \ldots v_{p_{m-1}} v_{p_m}\}$ and let the path $P_B = \{v_{2n} v_{2n+1} \ldots v_{4n} v_0\}$. Since $l(P_A) = diam(G)$, if a pebble were placed on a vertex in P_A, then it would be possible, with the x pebbles at v_{2n}, to move a pebble to v_{p_m}. Thus, no additional pebbles can be placed in P_A nor do we want any pebbles moved to P_A by placing pebbles on any vertex in P_B. Suppose some of the x pebbles at v_{2n} were used to move 1 pebble to each of the vertices in P_m (except v_{p_m}), then $2^{2n} - 1$ pebbles would remain at v_{2n}. Since $l(P_B) > diam(C_{4n+1})$, pebbles can be placed in the "middle" of P_B such that a pebble can not be moved to v_{2n} and, with the remianing pebbles from v_{2n}, not be able to move a (second) pebble to v_0. Since v_{3n} is n vertices from v_{2n} and v_{3n+1} is n vertices from v_0, y pebbles can be placed at v_{3n} and z pebbles can be placed at v_{3n+1} such that:

$$(1) \quad y + \tfrac{z}{2} < 2^n \qquad \text{and} \qquad (2) \quad \tfrac{y+2^n-1}{2} + z < 2^n$$

Since y and z are integers,

(3) $y + \frac{z}{2} \leq 2^n - \frac{1}{2}$ and (4) $\frac{y+2^n-1}{2} + z \leq 2^n - \frac{1}{2}$.

Solving for $y + z$,

(5) $y + z \leq 2^n - \frac{1}{3}$ implies (6) $y + z = \lfloor 2^n - \frac{1}{3} \rfloor = 2^n - 1$.

Since the pebbling number is one more than the maximum number of pebbles that can be placed on the graph such that a pebble could not be moved to every vertex in the graph, the pebbling number is $x + y + z + 1$. Therefore, $f(C_{4n+1} \cdot P_m) = 2^{2n+m} + 2^n - 1$. □

PROPOSITION 3

Let $n, m \in Z \geq 1$. Then, $f(C_{4n+3} \cdot P_m) = 2^{2n+m+1} + 2^n$

Proof.

Let $G = C_{4n+3} \cdot P_m$.

$V(G) = \{V(C_{4n+3})\} \cup \{V(P_m)\} = \{v_0, v_1, v_2, \ldots, v_{4n+2}\} \cup \{v_0, v_{p_1}, v_{p_2}, \ldots, v_{p_{m-1}}, v_{p_m}\}$.

$E(G) = \{E(C_{4n+3})\} \cup \{E(P_m)\} = \{v_0v_1, v_1v_2, \ldots, v_{4n+2}v_0\} \cup \{v_0v_{p_1}, v_{p_1}v_{p_2}, \ldots, v_{p_{m-1}}v_{p_m}\}$.

$diam(G) = d(v_{2n+1}, v_{p_m}) = 2n + m + 1$.

WLOG, we will designate v_{p_m} our target vertex.

Since $diam(G) = 2n+m+1$, $x = 2^{2n+m+1} - 1$ is the maximum number of pebbles that can be placed at v_{2n+1} such that a pebble could not be moved to v_{p_m}. Suppose x pebbles are placed at v_{2n+1}. Let the path $P_A = \{v_{2n+1}v_{2n} \ldots v_1v_0 \ldots v_{p_{m-1}}v_{p_m}\}$ and let the path $P_B = \{v_{2n+1}v_{2n+2} \ldots v_{4n+2}v_0\}$. Since $l(P_A) = diam(G)$, if a pebble were placed on a vertex in P_A, then it would be possible, with the x pebbles at v_{2n+1}, to move a pebble to v_{p_m}. Thus, no additional pebbles can be placed in P_A nor do we want any pebbles moved to P_A by placing pebbles on any vertex in P_B. Suppose some of the x pebbles at v_{2n+1} were used to move 1 pebble to each of the vertices in P_m (except v_{p_m}), then $2^{2n+1} - 1$ pebbles would remain at v_{2n+1}. Since $l(P_B) > diam(C_{4n+3})$, pebbles can be placed in the "middle" of P_B such that a pebble can not be moved to v_{2n+1} and, with the remianing pebbles from v_{2n+1}, not be able to move a (second) pebble to v_0. Since v_{3n+2} is $n + 1$ vertices from both v_{2n+1} and v_0, y pebbles can be placed at v_{3n+2} such that:

(1) $y < 2^{n+1}$ and (2) $y + 2^n - 1 < 2^{n+1}$.

Since $\forall n$, $y + 2^n - 1 \geq y$, we will solve for y from equation (2):

(3) $y < 2^n + 1$ implies (4) $y = 2^n$.

Since the pebbling number is one more than the maximum number of pebbles that can be placed on the graph such that a pebble could not be moved to every vertex in the graph, the pebbling number is $x + y + 1$. Therefore, $f(C_{4n+3} \cdot P_m) = 2^{2n+m+1} + 2^n$. □

DEFINITION ?(In progress)

A simple, connected graph, G, is called a *sun* if it contains exactly one cycle, C_k, and R distinct vertices not in C_k such that each $v_p \in R$ is connected to exactly one $v_i \in C_k$. A *sun* is denoted $C_k \cdot R_I^{(d)}$, where $i \in I$ iff $\{v_iv_p\} \in E(G)$ and $(d) = max\{d(i_a, i_b)\}$ $\forall i_a, i_b \in I$.

LEMMA ?

Suppose $n \geq 2$ and $R \geq 2$ are integers. If $d = n$ and $\forall i \in I$, then $f(C_{2n} \cdot R_I^{(d)}) = 2^{n+2} + R - 2$.

Proof.

Let $n \geq 2$ and $R \geq 2$ be integers. Let $G = C_{2n} \cdot R_I^{(d)}$ and let $d = n$. Then $V(G) = \{V(C_{2n})\} \cup \{V(R_I^{(d)})\} = \{v_0, v_1, v_2, \ldots, v_{2n-1}\} \cup \{v_j, v_s, v_t : 1 \leq j \leq R-2\}$, $E(G) = \{E(C_{2n})\} \cup \{E(R_I^{(d)})\} = \{v_0v_1, v_1v_2, \ldots, v_{2n-1}v_0\} \cup \{v_iv_j, v_nv_s, v_0v_t \ : \ v_i \in V(C_{2n})\}$. and $diam(G) = d(v_s, v_t) = n + 2$. WLOG, we will designate v_t our target vertex. Since $diam(G) = n + 2$, $x = 2^{n+2} - 1$ is the maximum number of pebbles that can be placed at v_s such that a pebble could not be moved to v_t. Suppose x pebbles are placed at v_s. Let the path $P_A = \{v_sv_nv_{n-1} \ldots v_1v_0v_t\}$ and let the path $P_B = \{v_sv_nv_{n+1} \ldots v_{2n-1}v_0v_t\}$. Since $l(P_A) = l(P_B) = diam(G)$, if a pebble were placed on a vertex in either P_A or P_B, then it would be possible with the x pebbles at v_s, to move a pebble to v_t via the path in which the pebble was placed. Thus, no additional pebbles can be placed in either P_A or P_B nor do we want a pebble moved to any vertex in either P_A or P_B from placing pebbles at any v_j. Hence, only one pebble can be placed at each v_j. Since there can be at most $(R-2)$ v_j's in G, the pebbling number is $x + R - 2 + 1$. Therefore, $f(C_{2n} \cdot R_I^{(d)}) = 2^{n+2} + R - 2$. □

PROPOSITION 4

Suppose $n \geq 1$ is an integer. Suppose $G = C_{4n} \cdot R_I^{(d)}$.

(*i*) if $d = 2n$ and $\forall i \in I$, then $f(G) = 2^{2n+2} + R - 2$.

(*ii*) if $d = 2n - 1$ $(n \geq 2)$ and $3n - 1, 3n \in I$, then $f(G) = 2^{2n+1} + \lfloor \frac{7(2^n)-2}{3} \rfloor + R - 2$,

(*iii*) if $d = 2n - 1$ $(n \geq 2)$ and $3n - 1 \in I$, $3n \notin I$, then $f(G) = 2^{2n+1} + 2^{n+1} + R - 3$,

(*iv*) if $d = 2n - 1$ $(n \geq 2)$ and $3n - 1 \notin I$, $3n \in I$, then $f(G) = 2^{2n+1} + 3(2^{n-1}) + R - 2$,

(*v*) if $d = 2n - 1$ $(n \geq 2)$ and $3n - 1, 3n \notin I$, then $f(G) = 2^{2n+1} + \lfloor \frac{7(2^{n-1})-1}{3} \rfloor + R - 2$,

(*vi*) if $d \leq 2n - 2$ and $\forall i \in I$, then $f(G) = 2^{2n+1} + R - 1$.

Proof. (In progress.)

In all cases, let $n \geq 1$ be an integer and let $G = C_{4n} \cdot R_I^{(d)}$.

Case (*i*)

Let $d = 2n$. Then, by LEMMA ?, $f(G) = 2^{2n+2} + R - 2 \ \forall i \in I$

In each of Case (*ii*) through Case (*v*), we will let $d = 2n - 1$ $(n \geq 2)$. WLOG, we will designate v_t our target vertex. Then, in each case, $diam(G) = d(v_{2n}, v_t) = d(v_s, v_t) = 2n+1$. Since $diam(G) = 2n+1$, $x = 2^{2n+1} - 1$ pebbles can be placed at either v_{2n} or v_s such that a pebble could not be moved to v_t. Suppose x pebbles were placed at v_{2n}. Let the path $P_A = \{v_{2n}v_{2n-1} \ldots v_1v_0v_t\}$ and let the path $P_B = \{v_{2n}v_{2n+1} \ldots v_{4n-1}v_0v_t\}$. Since $l(P_A) = l(P_B) = diam(G)$, we can use the same argument as in LEMMA ? and conclude that $2^{2n+1} + R - 1$ $(\forall i \in I)$ would be a sufficient number of pebbles in this configuration. However, in each case, placing x pebbles at v_s provides a worse configuration.

Case (*ii*)

Let $3n-1$, $3n \in I$. Suppose we placed x pebbles at v_s. Let the path $P_A = \{v_s v_{2n-1} v_{2n-2} \dots v_1 v_0 v_t\}$ and let the path $P_B = \{v_s v_{2n-1} v_{2n} \dots v_{4n-1} v_0 v_t\}$. Since $l(P_A) = diam(G)$, if a pebble were placed on a vertex in P_A, then it would be possible with the x pebbles at v_s, to move a pebble to v_t. Thus, no additional pebbles can be placed in P_A nor do we want a pebble moved to any vertex in P_A from placing pebbles in P_B.

PROPOSITION 5

$$
\begin{aligned}
&f(C_{4n+1} \cdot R_I^{(d)}) \\
&\quad = 2^{2n+2} + 2^{n+1} + R - 4 \text{ if } d = 2n \text{ for } 3n,\ 3n+1 \in I, \\
&\quad = 2^{2n+2} + 2^{n+1} + R - 5 \text{ if } d = 2n \text{ for } 3n \in I,\ 3n+1 \notin I, \\
&\quad = 2^{2n+2} + 2^{n} + R - 3 \text{ if } d = 2n \text{ for } 3n \notin I,\ 3n+1 \in I, \\
&\quad = 2^{2n+2} + R - 2 \text{ if } d = 2n \text{ for } 3n,\ 3n+1 \notin I, \\
&\quad = 2^{2n+1} + 2^{n+1} + R - 3 \text{ if } d = n+1 \text{ for } n \in I, \\
&\quad = 2^{2n+1} + 2^{n+1} + R - 4 \text{ if } d = n+1 \text{ for } n \notin I, \\
&\quad = 2^{2n+1} + 2^{n} + R - 2 \text{ if } d = n\ \forall i \in I \\
&\quad = 2^{2n+1} + R - 1 \text{ if } d \neq 2n, n+1, n \text{ for } 3n,\ 3n+1 \notin I.
\end{aligned}
$$

Proof. (In progress.)

PROPOSITION 6

$$
\begin{aligned}
&f(C_{4n+2} \cdot R_{\{i\}}^{(d)}) \\
&\quad = 2^{2n+3} + R - 2 \text{ if } d = 2n+1\ \forall i \in \{i\}, \\
&\quad = 2^{2n+2} + 3(2^{n}) + R - 2 \text{ if } d = 2n \text{ for } 3n+1 \in \{i\}, \\
&\quad = 2^{2n+2} + 3(2^{n-1}) + R - 2 \text{ if } d = 2n \text{ for } 3n+1 \notin \{i\}, \\
&\quad = 2^{2n+2} + R - 1 \text{ if } d \neq 2n+1, n+1\ \forall i \in \{i\}.
\end{aligned}
$$

Proof. (In progress.)

PROPOSITION 7

$$f(C_{4n+3} \cdot R_{\{i\}}^{(d)})$$

$= 2^{2n+3} + 2^{n+1} + R - 2$ if $d = 2n + 1$ for $3n + 2 \in \{i\}$,

$= 2^{2n+3} + 2^{n} + R - 2$ if $d = 2n + 1$ for $3n + 2 \notin \{i\}$,

$= 2^{2n+2} + 2^{n+1} + R - 1$ if $d = n + 1$ $\forall i \in \{i\}$,

$= 2^{2n+2} + 2^{n} + R - 1$ if $d \neq 2n + 1, n + 1$ $\forall i \in \{i\}$.

Proof.(In progress.)

PROPOSITION 8

$of(C_5 \times C_5) = 8$.

Proof. (In progress.)

PROPOSITION 9

Suppose $n \geq 3$ is an integer. If $r_1 = 1$, then $of(C_k) = of(C_k \cdot r_1 P_1) = \lceil \frac{2n}{3} \rceil = \lfloor \frac{2n+2}{3} \rfloor$.

Proof. (In progress.)

PROPOSITION 10

Suppose $n \geq 1$ is an integer. If $k = 3n + 2$, $r_1 = 2$, and $\forall d$, then $of(C_k \cdot r_1 P_1\{d, I\}) = 2n + 2$.

Proof. (In progress.)

PROPOSITION 11

Suppose $n \geq 0$ is an integer. If $k = 3n + 3$, $r_1 = 2$, and

if $3 \mid k$, then $of(C_k \cdot r_1 P_1\{d, I\}) = 2n + 2$,

if $3 \nmid k$, then $of(C_k \cdot r_1 P_1\{d, I\}) = 2n + 3$.

Proof. (In progress.)

PROPOSITION 12

Suppose $n \geq 0$ is an integer. If $k = 3n + 4$, $r_1 = 2$, and

if $3 \mid k$, then $of(C_k \cdot r_1 P_1\{d, I\}) = 2n + 3$,

if $3 \nmid k$, then $of(C_k \cdot r_1 P_1\{d, I\}) = 2n + 4$.

Proof. (In progress.)