Assignment 11

This assignment is due Friday, December 8, 2006 at 3 PM. It is worth 40 points.

In this assignment I use the notation $< x, y >$ in place of our textbook's $(x|y)$. (You may use either notation.) I also use $P_k(F)$ to denote the set of polynomials of degree k or less, over the field F.

1. Prove that if $<,>$ is an inner product on a vector space V and $< x, y > = < x, z > \ \forall x \in V$ then $y = z$.

2. (a) Given a subspace S, show that $S^{\perp}$ is a subspace and $V = S \oplus S^{\perp}$.

   (b) Given a vector y, define a function g from $V \to F$ by $g(x) := < x, y >$. Show this is a linear function.

   (c) Suppose V is of finite dimension with orthonormal basis B. Show that every linear functional f in V^* may be written as an inner product, that is, if $f \in V^*$ then there exists $y \in V$ such that $f(x) = < x, y >$ for all x.

3. For each of the following inner product spaces V and linear functionals $g : V \to F$, find a vector y such that $g(x) = < x, y >$ for all $x \in V$.

   (a) $V = \mathbb{C}^3, \ g(z_1, z_2, z_3) = z_1 + (2 + i)z_2 + (4 - 3i)z_3,$

   (b) $V = \mathbb{R}^3, \ g(x_1, x_2, x_3) = x_1 - 7x_2 + 5x_3.$

   (c) $V = \mathbb{C}^2, \ g(z_1, z_2) = z_1 - (2 - i)z_2,$

   (d) $V = P_1(\mathbb{R}), \ < f, h > := \displaystyle\int_0^1 f(t)h(t)dt, \ g(f) = f(1) + f'(1).$

   (e) $V = P_2(\mathbb{R}), \ < f, h > := \displaystyle\int_0^1 f(t)h(t)dt, \ g(f) = f(0) + f'(1).$

4. (a) Apply the Gram-Schmidt process to the vectors $\beta_1 = (1, 0, 1)$, $\beta_2 = (1, 0, -1)$, $\beta_3 = (0, 3, 4)$ to obtain an orthonormal basis for $\mathbb{R}^3$ with the standard inner product.

   (b) Let $V = \mathbb{R}^5$ with the standard inner product. Find an orthonormal basis for the subspace generated by $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$.

5. In each problem below, you are given a set S, an inner product space V and a vector $\vec{x}$. Apply the Gram-Schmidt process to S using the inner product of V. Then find the Fourier coefficients of $\vec{x}$.

   (a) $S = \{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$. $V = \mathbb{R}^3$, under the ordinary dot product. $\vec{x} = (1, 0, 1)$.

   (b) $S = \{(1, 2, 3), (1, -1, -2), (0, 0, 1)\}$. $V = \mathbb{R}^3$, under the ordinary dot product. $\vec{x} = (1, 0, 0)$.

   (c) $S = \{(1, 2, 2), (0, 1, -1), (1, 0, -2)\}$. $V = \mathbb{R}^3$, under the ordinary dot product. $\vec{x} = (1, 0, 1)$.

   (d) $S = \{t^2, t, 1\}$. $V = P_2(\mathbb{R})$. $< f, g > := \int_0^1 f(t)g(t)dt$. $x(t) = 1 + t$.

   (e) $S = \{t^2, 4t - 3, 1\}$. $V = P_2(\mathbb{R})$. $< f, g > := \int_0^1 f(t)g(t)dt$. $x(t) = 1$.