Review materials for the Final Exam, MTH 142, Spring 2008

The Final Exam is **Monday, May 12**, 8 - 10:20 AM in LDB 400.
Calculators will not be allowed on the Final Exam.
As always, you will be graded on your <u>work</u>.

Problems from Exams 2 and 3

1. Two boats meet in the middle of the ocean. At noon, one boat heads north at 30 miles per hour. The other boat stays at the meeting point for an extra two hours, then, at 2 PM, heads west at 20 miles per hour.
   (a) At 4 PM, what is the distance between the boats?
   (b) At 4 PM, how rapidly is the distance between the boats changing? (Show your work/explain your reasoning.)

2. The graph of $y = f(x)$ is given below. Using that graph, draw in the space provided below it, the graph of the first derivative. Label any critical points.

3. Suppose
$$f(x) = x^4 - 4x^3 + 4x^2 + 18.$$
   (a) Use the first derivative to find the critical points of the graph of $y = f(x)$ and determine which are local minima and which are local maximum.
   (b) Use the second derivative to find inflection points of the graph of $y = f(x)$.
   (c) Determine where the graph of $y = f(x)$ is concave up and concave down.
   (d) On the additional paper provided, carefully draw a graph of the curve $y = f(x)$. Label all points found above.

4. Find $\dfrac{dy}{dx}$.
   (a) $\sin y + x^2 y = \pi e^2$.
   (b) $y^3 + xy^2 = \cos x$.
   (c) $y = \ln(x^3 + 5x)$
   (d) $y = \ln(\sec x)$

5. Compute $\dfrac{dy}{dx}$. Put your answer in terms of x *if* possible.
   (a) $\ln y = x \ln x$
   (b) $\sin y = x$

6. A girl flies a kite at a height of 50 meters. The wind carries her kite horizontally away from her at a rate of 10 meters per second. How fast must she let out the string when the kite is 130 meters away from her?

7. (a) i. Find the equation for the line through $(0, 1)$ with slope 1.
   ii. Find the equation for the line through $(4, 2)$ with slope $\frac{1}{4}$.
   (b) Linearize each of the functions $f(x)$, below, at the point $(a, f(a))$.
   i. $f(x) = e^x$, $a = 0$.
   ii. $f(x) = \sqrt{x}$, $a = 4$.
   (c) Use your work in part (b), above, to estimate the values of:
   i. $e^{0.01}$.
   ii. $\sqrt{3.9}$.

8. Suppose x is a positive real number.
   (a) Find a formula for $\sec(\tan^{-1}(x))$.
   (b) Find a relationship between $\tan^{-1}(x)$ and $\cot^{-1}(x)$.

9. In this problem you will model changes in a certain spherical hailstone bobbing up and down in a thunderhead. *Please make sure your answers have correct units.*
   (a) Suppose the hailstone has radius 5 mm. As the hailstone rises in the cloud, another layer of ice is added to the surface of the hailstone, increasing its radius by 0.2 mm. Use differentials to estimate the change in volume of the hailstone due to this increase.
   (b) Assuming that the hailstone has radius 5 mm. and that the radius is increasing by 0.2 mm./sec., how rapidly is the volume increasing?

10. A twenty foot ladder is leaning against a wall. The base of the ladder slides away from the wall at 2 feet per second. How rapidly is the height of the ladder changing when the base is 12 feet away from the wall?

11. Suppose

$$f(x) = \frac{(x+1)^2}{1+x^2}.$$

The first and second derivatives of $f(x)$ are

$$f'(x) = \frac{2(1-x^2)}{(1+x^2)^2} \quad \text{and} \quad f''(x) = \frac{4x(x^2-3)}{(1+x^2)^3}.$$

   (a) Use the first derivative (given above) to find the critical points of the graph of $y = f(x)$ and determine which are local minima and which are local maximum.
   (b) Use the second derivative (given above) to find inflection points of the graph of $y = f(x)$.
   (c) Determine where the graph of $y = f(x)$ is concave up and concave down.

12. What are the dimensions of the lightest open-top cylindrical can that will hold a volume of 1000 cm^3?

13. Linearize the function $f(x) = x^3 - 5$ at the point
   (a) where $x = 2$,
   (b) where $x = 1$.
   (c) Take the line in your answer in part (a) and find the x-intercept of that line.
   (d) Use Newton's method to estimate the solution to $x^3 - 5 = 0$. Start with $x_0 = 2$ and find x_2. (Use the back or extra paper if necessary.)
   (e) What is the exact solution to the equation $x^3 - 5 = 0$?

Sample Final Exam

0. Find *all* x such that $\sec x = -\sqrt{2}$.

1. Compute the following limit $\lim_{x \to 2} \frac{x^3 - 8}{x^2 + 3x - 10}$.

2. Compute $\frac{dy}{dx}$.

   (a) $y = 5x^9 - 4\sqrt{x} + \frac{1}{\sqrt[3]{x}} + 19e^x - \frac{\pi}{x}$.

   (b) $y = x^2 \cos x$.

   (c) $y = \frac{\sin x}{x^2 + 5x}$.

   (d) $y = e^{\tan(x^2)}$

   (e) $y^2 + x\sin(y) = e^{6x}$.

   (f) $x = e^y$.

   (g) $y = s^2 t$ where s and t are functions of x.

   (h) $y = 2\pi r^2 + r^2 h$ where r and h are functions of x.

3. (a) Find the equation for the tangent line to the curve at $f(x) = x^3 - 30$ at the point where $x = 3$.

   (b) Use this to approximate $f(3.2)$.

   (c) Use Newton's method with first guess $x_1 = 3$ to find a second approximation x_2 to a solution to the equation $x^3 = 30$.

4. A graph has the following properties:

   (a) $f(-4) = 0$; $f'(-4)$ is undefined;

   (b) $f(2) = 0$, $f'(2) = -5$;

   (c) $f(6) = -4$; $f'(6) = 3$; $f''(6) = 0$;

   (d) $f'(4) = 0$.

   (e) $f''(x) > 0$ if x is in the interval $(0, 6)$;

   (f) $f''(x)$ is negative in the region $(-\infty, -4) \cup (-4, 0) \cup (6, \infty)$;

   (g) $\lim_{x \to 0+} f(x) = \infty$ and $\lim_{x \to 0^-} f(x) = -\infty$,

   (h) $\lim_{x \to \infty} f(x) = -1$.

   (i) $\lim_{x \to -\infty} f(x) = 2$.

   The graph is continuous everywhere except when $x = 0$. Carefully draw a picture of the graph. Then write out the equations for any asymptotes.

5. Consider the function $f(x) = \dfrac{x}{e^x}$.

   (a) Give the domain of the function. Give the intercepts. Describe any symmetry.

   (b) Use the first derivative to find the exact value of the critical points. Use the second derivative test to determine if these critical points are maxima or minima.

   (c) Use the second derivative to find any inflection points.

   (d) Give the intervals where the function is increasing or decreasing.

   (e) Give the intervals where the function is concave up or down.

   (f) Draw a graph of the function, labeling the points you have found above.

6. We have \$1000 to spend in building a rectangular pen. We need to reinforce the walls on the south and north sides, so the cost per foot of fencing on those sides is \$5 per foot. The walls on the west and east sides will be cheaper, costing only \$2 per foot. Give the dimensions which maximize the area of the pen.

   Use the second derivative test to prove that your answer is a maximum.

7. A rectangular poster is to have a rectangular inner printed area of 100 in^2. The poster will have 3 inch margins at the bottom and top and 1 inch margins on the sides. Find the dimensions of the poster which minimize the total area of the poster.

   Use the second derivative test to prove that your answer is a minimum.

8. Use Riemann sums with $n = 2$ rectangles to estimate $\displaystyle\int_1^9 x^3\,dx$, using the

   (a) left endpoints,

   (b) right endpoints,

   (c) midpoints.

9. Compute the following sums:

   (a) $\displaystyle\sum_{i=2}^{4} \frac{1}{i^2}$.

   (b) $\displaystyle\sum_{i=0}^{3} i^3$.

   (c) $\displaystyle\sum_{i=1}^{3} \sin(\frac{\pi i}{2})$.

10. Evaluate the following integrals. (If you evaluate an integral using the method of substitution then you must write out both u and du.)

(a) $\displaystyle\int_1^9 \frac{1}{\sqrt{w}}\, dw$

(b) $\displaystyle\int_{\pi/6}^{\pi/2} \cos s\, ds$

(c) $\displaystyle\int_{e^2}^{e^4} \frac{1}{t}\, dt$

(d) $\displaystyle\int_1^x 5t^2\, dt$

(e) $\displaystyle\int_1^3 \frac{x\, dx}{\sqrt{x^2+3}}$

(f) $\displaystyle\int (2-\sqrt{w})^2\, dw$

(g) $\displaystyle\int \frac{y+4y^3}{y^4}\, dy$

(h) $\displaystyle\int \frac{1}{1-\sin^2(\theta)}\, d\theta$

(i) $\displaystyle\int \frac{x^2}{(x^3+5)^2}\, dx$

(j) $\displaystyle\int \frac{x^2}{x^3+5k}\, dx$

Final Exam, MTH 142, Fall 2007

1. (8 percent) Give the definition of the derivative. Then *explain* how the definition was created. (Your explanation should correctly use terms such as "secant line", etc.) Finally, demonstrate the use of the definition by showing that the derivative of $y = x^3$ is $3x^2$.

2. (8 %) Compute the following limits. Describe the method you use.

   (a) $\lim_{x \to 2} \frac{x^2 - 4}{x^2 + 3x - 10}$.

   (b) $\lim_{x \to \infty} \frac{x^2 + x}{e^x + x}$.

3. (8 %)

   (a) Find the equation for the tangent line to the curve $f(x) = \sqrt{x}$ at the point $(25, 5)$.

   (b) Use the tangent line in (a) to approximate the following to two decimal places:

      i. $\sqrt{25.1}$,

      ii. $\sqrt{24.9}$

4. (20 %) Compute $\frac{dy}{dx}$.

   (a) $y = (\sin x + 5e^x)^{200}$.

   (b) $y = \frac{\ln x}{\sqrt{x}}$.

   (c) $y = \ln(\sec x)$.

   (d) $x^3 + y^3 + \sin y = 4x - 8$.

   (e) $y = u^2 + \sin u$ (where u is a function of x.)

5. (10 %) Consider the function

$$f(x) = x^3 + 2x^2 + x + 2.$$

   (a) Use first derivative to find the exact value of the critical points. Determine if these critical points are maxima or minima.

   (b) Use the second derivative to find all inflection points.

   (c) Give the intervals where the function is increasing or decreasing.

   (d) Give the intervals where the function is concave up or down.

   (e) On a separate pink sheet (provided), sketch a graph the curve and label the points you have found, above.

6. (8 %) A young boy, a kite-runner, is flying his kite at a constant height of 30 meters. A strong wind threatens to pull his kite away from him. The wind out of the north is pushing his kite horizontally at 5 meters per second. We take a snapshot of this system when the kite is directly above a point 40 meters south of the kite-runner and we ask the following questions. (See the figure.)

   (a) What is the relationship between the length of the kite string and the horizontal displacement, x, of the kite?
   (b) What is the relationship between the horizontal displacement of the kite and the angle, θ, that the kite string makes with the ground?
   (c) At the moment of our snapshot, how fast is the kite string playing out?
   (d) At the moment of our snapshot, what is the rate of change of the angle θ?

7. (10 %) A metal box has a square base and no top. The material used to make the four sides and the bottom of the box comes to 75 square inches. Find the dimensions of the box which maximize the volume. (Use the second derivative test to show that your answer is indeed a maximum.)

8. (6 %)

   (a) Compute the sum $\sum_{i=0}^{4} i^2$.
   (b) Use Riemann sums with $n = 2$ rectangles to estimate $\int_0^4 x^2\ dx$, using the
      i. left endpoints,
      ii. right endpoints,
   (c) What is the exact value of $\int_0^4 x^2\ dx$?

9. (20 %) Evaluate the following integrals. (If you do a substitution, you must show both u and du.)

   (a) $\int x^3 + \sqrt{x} + \frac{1}{\sqrt{x}} + \frac{1}{x}\ dx$
   (b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ dx$
   (c) $\int (x+1)(x^3+3x+7)^5\ dx$
   (d) $\int \frac{\sin x}{\cos x + 4}\ dx$
   (e) $\int_1^2 \frac{x-1}{(x^2-2x+4)^2}\ dx$

10. (5 %) Multiple choice. Circle the *best* answer.

(a) The trigonometric functions arise from the study of

i. the unit circle,
ii. the triangle,
iii. the square,
iv. the parabola,
v. the earth.

(b) The inverse of a function is the function (or relation) created by

i. taking the reciprocal,
ii. swapping the domain and range,
iii. changing every x to a y,
iv. subtracting the function,
v. using $1/x$ in place of x.

(c) The derivative of a function at a point on the curve is

i. obtained by changing x^n into nx^{n-1}.
ii. the slope of the secant line at the point,
iii. the slope of the tangent line at the point,
iv. the maximum or minimum of the function,
v. a concept invented by the Babylonians.

(d) A function $f(x)$ has a limit L at a point $x = a$ if

i. we can plug a into the function and get L,
ii. we can plug L into the function and get a,
iii. when we get very close to L then the function is always very close to a,
iv. when we get very close to a then the function is always very close to L,
v. $f(x)$ is a trig function.

(e) The Fundamental Theorem of Calculus states that

i. derivatives are the fundamental tool used in the sciences,
ii. the derivative of an exponential function is an exponential function,
iii. we can create new functions by considering the area under the curve $y = f(t)$ (over some region $[a, x]$),
iv. If a function is created by computing the area "under" the curve $y = f(t)$, then the integral of that new function is $f(x)$.
v. If a function is created by computing the area "under" the curve $y = f(t)$, then the derivative of that new function is $f(x)$.