5.3 Triangulation and the Law of Sines

A triangle has three sides and three angles; their magnitudes provide six pieces of information about the triangle. In ordinary (Euclidean) geometry, *most* of the time three pieces of information are sufficient to give us the other three pieces of information.

In order to more easily discuss the angles and sides of a triangle, we will label the angles by capital letters (such as \( A, B, C \)) and label the sides by small letters (such as \( a, b, c \)). Furthermore, we will assume that a side labeled with a small letter is the side opposite the angle with the same, but capitalized, letter. For example, the side \( a \) is opposite the angle \( A \). We will also use these labels for the lengths (magnitudes) of the angles and sides, so that \( a = 3 \) is a reasonable abbreviation for the statement “side \( a \) has length 3.”

5.3.1 Three Bits of a Triangle

Suppose we have three bits of information about a triangle. Can we recover all the information? If the three bits of information are the values of the three angles, and if we have no information about lengths of sides, then the answer is *No*. Given a triangle with angles \( A, B, C \) (in Euclidean geometry) we can always expand (or contract) the triangle to a *similar* triangle which has the same angles but whose sides have all been stretched (or shrunk) by some constant factor. This “Angle-Angle-Angle” information (abbreviated *AAA*) is *not* enough information to describe the entire triangle. However, we will discover that in almost every other situation, we can recover the entire triangle.

If we know the values of two angles and one side, then, if the known side is between the two angles, we will say that we are in the Angle-Side-Angle case. (We abbreviate this case *ASA*.) If we know two angles and one side, but the known side is *not* between the two angles, then we are in the Angle-Angle-Side (AAS) situation and we can also recover the remaining bits of the triangle.

Indeed, any time we know two angles then, since the sum of the angles of a triangle is 180°, we really know all angles.

In the next section we will describe how to solve the *ASA* and *AAS* cases.

5.3.2 The Law of Sines

The law of sines says that give a triangle \( \triangle ABC \), with lengths of sides \( a, b \) and \( c \) then (with the understanding that capital letters are angles opposite sides represented by little letters)

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]  

(23)

Why is this true? Start with a triangle with vertices \( A, B, C \) (drawn with acute angles at \( A \) and \( B \) in Figure 3.) Draw a perpendicular from the vertex \( C \) onto the side \( AB \) and let \( D \) represent the right angle formed there.

![Figure 3.](image-url)
If \( h \) is the height of the triangle then

\[
\sin A = \frac{h}{b} \quad \text{and} \quad \sin B = \frac{h}{a}.
\]

Let’s solve for \( h \):

\[
h = b \sin A \quad \text{and} \quad h = a \sin B.
\]

Since both \( b \sin A \) and \( a \sin B \) are equal to \( h \) then

\[
b \sin A = a \sin B.
\]

Dividing both sides by \( ab \) we have

\[
\frac{\sin A}{a} = \frac{\sin B}{b}.
\]

A similar argument tells us that

\[
\frac{\sin A}{a} = \frac{\sin C}{c}
\]

and so the ratio of the sine of the angle to the opposite side is a constant (an “invariant”) of the triangle. (Remark: if the angle \( A \) or \( B \) is obtuse then the picture drawn in Figure 3 will change but the computations will not. Our result does not depend on the drawing of Figure 3.)

**5.3.3 The area of an oblique triangle**

An “oblique” triangle is one which is not a right triangle. Before we move on from Figure 3 and our proof of the Law of Sines, we pause to make an observation about the area of a triangle.

Since a triangle is half of a parallelogram, its area is one-half of the product of its base and height. We let \( K \) represent the area of a triangle (since we are already using the letter \( A \) for an angle.) Looking at Figure 3, we see that we could write \( c \) for the base of the triangle and so the area of the triangle is

\[
K = \frac{1}{2} ch.
\]

But earlier, in our proof of the Law of Sines, we solved for \( h \) and we wrote \( h = b \sin A \) and \( h = a \sin B \). So we can substitute for \( h \) and write the area as

\[
K = \frac{1}{2} cb \sin A \quad \text{or} \quad K = \frac{1}{2} ac \sin B
\]

Or we could call the known angle \( C \) and just write

\[
K = \frac{1}{2} ab \sin C
\] (24)

We can summarize this by saying that the area of a triangle is one-half of the product of the sine of an angle and its neighboring sides.

**5.3.4 Two Angles and the Law of Sines**

If we know two angles of a triangle, then since the three angles add to \( 180^\circ = \pi \) then we can figure out the third angle. As long as we know one more bit of information, the length of a side, then the law of sines gives us the length of all sides. In our abbreviated notation for the known information, this includes the cases \( ASA \) and \( AAS \), that is, the cases in which we know two angles and an included side or two angles and a side not between them. These cases are equivalent since we know all three angles once we know any two.

**Some Worked Problems.**

Use the Law of Sines to solve the following triangles.
1. \( A = 62^\circ, B = 74^\circ, c = 14 \text{ feet} \)

**Solution.** If \( A = 62^\circ, B = 74^\circ, c = 14 \text{ feet} \) then \( C = 54^\circ \). Then, by the law of sines,

\[
\frac{\sin 54^\circ}{14} = \frac{\sin 62^\circ}{a} = \frac{\sin 75^\circ}{b}
\]

so

\[
a = \frac{14 \sin 62^\circ}{\sin 54^\circ} \approx 15.28 \text{ feet and } b = \frac{14 \sin 75^\circ}{\sin 54^\circ} \approx 16.72 \text{ feet.}
\]

2. Solve the triangle \( A = 60^\circ, B = 70^\circ, c = 10 \text{ feet} \)

**Solution.** If \( A = 60^\circ, B = 70^\circ, c = 10 \text{ feet} \) then \( C = 50^\circ \) since the sum of the angles of a triangle is \( 180^\circ \). By the law of sines,

\[
\frac{\sin 60^\circ}{a} = \frac{\sin 70^\circ}{b} = \frac{\sin 50^\circ}{10}
\]

So

\[
a = \frac{10 \sin 60^\circ}{\sin 50^\circ} = 11.31 \text{ feet and } b = \frac{10 \sin 70^\circ}{\sin 50^\circ} = 12.27 \text{ feet}
\]

3. \( a = 10, A = 30^\circ, B = 50^\circ \)

**Solution.** Since angles sum to \( 180^\circ \) then \( C = 100^\circ \). By the Law of Sines

\[
a = 10, \ b = 15.32, \ c = 19.70, \ A = 30^\circ, \ B = 50^\circ, \ C = 100^\circ.
\]

4. \( c = 10, A = 30^\circ, B = 50^\circ \).

**Solution.** Since angles sum to \( 180^\circ \) then \( C = 100^\circ \). By the Law of Sines

\[
a = 5.08, \ b = 7.78, \ c = 10, \ A = 30^\circ, \ B = 50^\circ, \ C = 100^\circ.
\]

5.3.5 **The law of sines and SSA**

If we know only one angle of a triangle but two sides, sometimes the Law of Sines is sufficient. To use the laws of sines, we need one of the sides to be opposite the known angle. In this case, our information is often abbreviated SSA – we know two sides and then an angle *not* between them.

We do have to be a bit careful here. Just because we know the sine of an angle does not mean we know the value of the angle.

**Some Worked Problems.**

Solve the following triangles. (Find *all* sides and all angles.)

1. \( A = 30^\circ, a = 6 \text{ feet}, b = 8 \text{ feet} \)

**Solution.** To solve \( A = 30^\circ, a = 6 \text{ feet}, b = 8 \text{ feet} \) apply the law of sines:

\[
\frac{\sin 30^\circ}{6} = \frac{\sin B}{8} = \frac{\sin C}{c}
\]
This forces
\[ \sin B = \frac{8 \sin 30^\circ}{6} = \frac{2}{3} \]
so \( B \) is either \( \sin^{-1} \left( \frac{2}{3} \right) \approx 41.81^\circ \) or \( 180^\circ - 41.81^\circ = 138.19^\circ \).

If \( B = 41.81^\circ \) then \( C = 108.19^\circ \). If \( B = 138.19^\circ \) then \( C = 11.81^\circ \).

By the law of sines
\[ c = \frac{6 \sin C}{\sin 30^\circ} = 12 \sin C. \]
Now if \( C = 108.19^\circ \) then
\[ c = 12 \sin 108.19^\circ = 11.40 \]
If \( C = 11.81^\circ \) then
\[ c = 12 \sin 11.81^\circ = 2.46 \]

There are two solutions, two triangles:

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2. \( A = 60^\circ \), \( a = 10 \) feet, \( c = 12 \) feet

**Solutions.** If \( A = 60^\circ \), \( a = 10 \) feet, \( c = 12 \) feet then by the law of sines
\[ \frac{\sin 60^\circ}{10} = \frac{\sin C^\circ}{12} \]
So
\[ \sin C = \frac{12 \sin 60^\circ}{10} = \frac{3\sqrt{3}}{5} \approx 1.039 > 1. \]

There is no angle with sine greater than 1 so there is no solution; **this triangle cannot exist**.

3. \( a = 10 \), \( b = 6 \), \( B = 20^\circ \).

**Solution.** Use the Law of Sines to see that \( A \) could be either \( A = 34.75^\circ \) or \( A = 145.25^\circ \), Then \( C = 125.25^\circ \) or \( C = 14.75^\circ \). Finally use the Law of Sines to finish off the problem.

There are **two answers**. Both must be listed.

\[ a = 10, \ b = 6, \ c = 14.33, \ A = 34.75^\circ, \ B = 20^\circ, \ C = 125.25^\circ \]

and

\[ a = 10, \ b = 6, \ c = 4.47, \ A = 145.25^\circ, \ B = 20^\circ, \ C = 14.75^\circ. \]

4. \( b = 12 \), \( c = 10 \), \( C = 60^\circ \).

**Solution.** By the Law of Sines, \( \sin B = \frac{12}{10} (\sin 60^\circ) = \frac{3\sqrt{3}}{5} \approx 1.039. \) Since the sine of \( B \) cannot be greater than one, **no such triangle is possible**. (The length of \( b \) is not large enough to “reach” the side \( a \).)
5. \( b = 12, c = 10, C = 45^\circ \).

**Solution.** By the Law of Sines either \( B \approx 58.06^\circ \) or \( B \approx 121.95^\circ \). Then \( A \approx 76.95^\circ \) or \( A \approx 13.05^\circ \).

Finally use the Law of Sines to finish off the problem.

There are two answers. Both must be listed.

\[
\begin{align*}
a &= 13.78, & b &= 12, & c &= 10, & A &= 76.95^\circ, & B &= 58.45^\circ, & C &= 45^\circ, \\
a &= 3.19, & b &= 12, & c &= 10, & A &= 13.05^\circ, & B &= 121.95^\circ, & C &= 45^\circ
\end{align*}
\]

### 5.3.6 Triangulation

One can often measure distance to an object by working out a baseline and then measuring the angles formed by lines from the distant object to the ends of the baseline. This is a classic case of ASA and is perfect for the law of sines.

**A Worked Problem**

Suppose that a ship is out in the harbor. Standing at the dock, a compass reveals that the ship is at a bearing of 30\(^\circ\) east of north. But if one walks 200 yards west of the dock, the ship is now at a bearing of 40\(^\circ\) east of north. How far is the ship from the dock?

If we draw an east-west baseline representing the 200 yards walked, and then draw lines from the ends of the baseline to the ship, we have a triangle in which the vertex on the west side of the baseline has an (interior) angle of 90\(^\circ\) − 40\(^\circ\) = 50\(^\circ\) and the vertex on the east side of the baseline has an angle of 90\(^\circ\) + 30\(^\circ\) = 120\(^\circ\) and so, if we call the length of the baseline \( c = 200 \), we have angles \( A = 50^\circ \) and \( B = 120^\circ \). The vertex faraway at the ship has angle 10\(^\circ\) in this triangle (this is the difference in 40\(^\circ\) and 30\(^\circ\).)

Let us use \( C \) to represent the angle 10\(^\circ\) at the vertex of this triangle where the ship sits. Then the side \( c \) is the 200 yard baseline and the angles \( A \) and \( B \) are 50\(^\circ\) and \( B = 120^\circ \).

We can solve this problem by the Law of Sines and see that \( a = 882, b = 997.45 \) So the ship is 882 yards from the east end of the dock.
5.3.7 Other resources on trig identities and trig equations

In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at stitz-zeager.com) this material is covered in section 11.2 and 11.3.


Here are some online resources:

1. Khan Academy videos on trig identities
2. Dr. Paul's online math notes include a review of trig formulas and a guide to solving trig equations

(Remember: the philosophy of this class is “understand; don’t memorize!”)

**Worksheet to go with these notes.**

As class homework, please complete *Worksheet 5.3, Law of Sines and Triangulation*, available through the class webpage.