Elementary Functions
Part 4, Trigonometry
Lecture 4.7a, Solving Problems with Inverse Trig Functions

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**Invers trig functions create right triangles**

An inverse trig function has an angle \((y \text{ or } \theta)\) as its output. That angle satisfies a certain trig expression and so we can draw a right triangle that represents that expression.

*One can always draw a right triangle with an inverse trig function and think of the output as a certain angle in that triangle.*

For example, the equation \(\arcsin(z) = \theta\) implies that \(\sin \theta = z\) and so corresponds to a right triangle with hypotenuse 1, with \(\theta\) one of the acute angles and \(z\) the length of the side opposite \(\theta\).

![Diagram of a right triangle with \(\theta\) as one of the acute angles and \(z\) as the length of the side opposite \(\theta\).]

Some Worked Problems on Inverse Trig Functions

We will practice this idea with some worked problems....

1. Draw a right triangle with the appropriate lengths and use that triangle to find the sine of the angle \(\theta\) if
   1. \(\cos(\theta) = \frac{2}{3}\)
   2. \(\cos(\theta) = \frac{4}{5}\)
   3. \(\cos(\theta) = 0.8\).
   4. \(\cos(\theta) = 0.6\).

**Partial solutions.**

1. If \(\cos(\theta) = \frac{2}{3}\) then draw a triangle with legs of length 2, \(\sqrt{5}\) and hypotenuse of length 3. If the cosine of \(\theta\) is \(\frac{2}{3}\) then the sine of \(\theta\) is \(\frac{\sqrt{5}}{3}\).
2. If \(\cos(\theta) = \frac{2}{5}\) then draw a triangle with legs of length 2, \(\sqrt{21}\) and hypotenuse of length 5. The sine of \(\theta\) is \(\frac{\sqrt{21}}{5}\).
3. If \(\cos(\theta) = 0.8\) then draw a triangle with legs of length 3, 4 and hypotenuse of length 5. The sine of \(\theta\) is \(\frac{3}{5}\).
4. If \(\cos(\theta) = 0.6\) then draw a triangle with legs of length 3, 4 and hypotenuse of length 5. The sine of \(\theta\) is \(\frac{4}{5}\).

Some Worked Problems on Inverse Trig Functions

When we work with inverse trig functions it is especially important to draw a triangle since the output of the inverse trig function is an angle of a right triangle.

Indeed, one could think of inverse trig functions as “creating” right triangles. The angle \(\theta\) in the drawing below is \(\arcsin(z)\). Notice that the Pythagorean theorem then gives us the third side of the triangle (written in blue); its length is \(\sqrt{1 - z^2}\). This allows us to simplify expressions like \(\cos(\arcsin z)\), recognizing that

\[
\cos(\arcsin z) = \cos(\theta) = \sqrt{1 - z^2}.
\]

In a similar manner, we can simplify \(\tan(\arcsin z)\) to

\[
\tan(\arcsin(z)) = \frac{z}{\sqrt{1 - z^2}}.
\]
Some Worked Problems on Inverse Trig Functions

Simplify (without use of a calculator) the following expressions

1. \( \text{arcsin}[\sin(\frac{\pi}{8})] \).
2. \( \text{arccos}[\sin(\frac{\pi}{8})] \).
3. \( \cos[\text{arcsin}(\frac{1}{3})] \).

Solutions.

1. Since \( \text{arcsin} \) is the inverse function of sine then \( \text{arcsin}[\sin(\frac{\pi}{8})] = \frac{\pi}{8} \).
2. If \( \theta \) is the angle \( \frac{\pi}{8} \) then the sine of \( \theta \) is the cosine of the complementary angle \( \frac{\pi}{2} - \frac{\pi}{8} \), which, after getting a common denominator, simplifies to \( \frac{3\pi}{8} \). In other words, the sine of \( \frac{\pi}{8} \) is the cosine of \( \frac{3\pi}{8} \). So \( \text{arccos}[\sin(\frac{\pi}{8})] = \frac{3\pi}{8} \). (Notice that I’ve solved this problem this without ever having to figure out the value of \( \sin(\frac{\pi}{8}) \).)
3. To simplify \( \cos[\text{arcsin}(\frac{1}{3})] \) we draw a triangle with hypotenuse of length 3 and one side of length 1, placing the angle \( \theta \) so that \( \sin(\theta) = \frac{1}{3} \). The other short side of the triangle must have length \( \sqrt{8} = 2\sqrt{2} \) by the Pythagorean theorem so the cosine of \( \theta \) is \( \frac{2\sqrt{2}}{3} \).

So \( \cos[\text{arcsin}(\frac{1}{3})] = \frac{2\sqrt{2}}{3} \).

Some worked problems.

5. Simplify \( \text{arccos}(y) + \text{arcsin}(y) \).

Solution. Notice in the triangle in the figure below, that the sine of \( \theta \) is \( y \) and the cosine of \( \frac{\pi}{2} - \theta \) is \( y \).

So \( \text{arcsin}(y) = \theta \) and \( \text{arccos}(y) = \frac{\pi}{2} - \theta \). Therefore

\[ \text{arccos}(y) + \text{arcsin}(y) = \theta + (\frac{\pi}{2} - \theta) = \frac{\pi}{2} \].

Indeed, the expression \( \text{arccos}(y) + \text{arcsin}(y) \) merely asks for the sum of two complementary angles! By definition, the sum of two complementary angles is \( \frac{\pi}{2} \!).
Drawing triangles to solve composite trig expressions

Some problems involving inverse trig functions include the composition of the inverse trig function with a trig function. If the inverse trig function occurs first in the composition, we can simplify the expression by drawing a triangle.

**Worked problems.** Do the following problems *without* a calculator. Find the exact value of

1. \( \sin(\arccos(-\frac{3}{4})) \)
2. \( \tan(\arcsin(-\frac{3}{4})) \)

**Solutions.**

1. To compute \( \sin(\cos^{-1}(-\frac{3}{4})) \) draw a triangle with legs 3, \( \sqrt{7} \) and hypotenuse 4. The angle \( \theta \) needs to be in the second quadrant so the cosine will be negative. In this case, the sine will be positive. So the sine of the angle \( \theta \) should be \( \frac{\sqrt{7}}{4} \).

2. To compute \( \tan(\sin^{-1}(-\frac{3}{4})) \) draw a triangle with legs 3, \( \sqrt{7} \) and hypotenuse 4. The tangent of the angle \( \theta \) should be \( \frac{3}{\sqrt{7}} \). But the angle \( \theta \) is in the fourth quadrant so the final answer is \( -\frac{3}{\sqrt{7}} \).

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**Drawing triangles to solve composite trig expressions**

Simplify \( \sin(2 \arctan(-\frac{4}{3})) \) (Use the trig identity \( \sin 2\theta = 2 \sin \theta \cos \theta \).)

**Solution.** To compute \( \sin(2 \tan^{-1}(-\frac{4}{3})) = 2 \sin \theta \cos \theta \) where \( \tan(\theta) = -\frac{4}{3} \) draw a triangle with legs 3, 4 and hypotenuse 5. The cosine of the angle \( \theta \) is \( \frac{3}{5} \) and the sine of the angle \( \theta \) is \( -\frac{4}{5} \). Since the original problem has a negative sign in it, and we are working with the arctangent function, then we must be working with an angle in the fourth quadrant, so the sine is really \( -\frac{4}{5} \). Now we just plug these values into the “magical” identity given us:

\[
\sin(2\theta) = 2 \sin \theta \cos \theta = 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) = -\frac{24}{25}
\]

Simplify the following expressions involving arctangent:
\( \tan(\arctan(z)) \), \( \sin(\arctan(z)) \), \( \cot(\arctan(z)) \), \( \sec(\arctan(z)) \).

**Solutions.**

1. To compute \( \tan(\arctan(z)) \) just recognize that \( \tan z \) and \( \arctan z \) are inverse functions and so \( \tan(\arctan(z)) = z \).

2. To compute \( \sin(\arctan(z)) \) draw a right triangle with sides 1, \( z \) and hypotenuse \( \sqrt{1+z^2} \). The sine of the angle \( \theta \) is \( \frac{z}{\sqrt{1+z^2}} \).

3. In the figure above, the cotangent of the angle \( \theta \) is \( \frac{1}{z} \).

4. The secant of the angle \( \theta \) should be \( \frac{1}{\cos \theta} = \frac{1}{\sqrt{1+z^2}} \).
More on inverting composite trig functions

Just like other functions, we can algebraically manipulate expressions to create an inverse function.

**Some worked problems.** Find the inverse function of \( y = \sin(\sqrt{x}) + 2 \)

**Solutions.** To find the inverse function of \( y = \sin(\sqrt{x}) + 2 \), let’s exchange inputs and outputs:

\[
x = \sin(\sqrt{y}) + 2
\]

and then solve for \( y \) by subtracting 2 from both sides

\[
x - 2 = \sin(\sqrt{y}),
\]

applying the arcsin to both sides,

\[
\arcsin(x - 2) = \sqrt{y}
\]

and then squaring both sides

\[
(arcsin(x - 2))^2 = y
\]

so that the answer is \( y = (arcsin(x - 2))^2 \).

Find the inverse function of \( y = \sin(\sqrt{x + 2}) \)

**Solutions.** We set

\[
x = \sin(\sqrt{y} + 2),
\]

take the arcsine of both sides:

\[
\arcsin(x) = \sqrt{y + 2},
\]

square both sides

\[
(arcsin(x))^2 = y + 2,
\]

and then subtract 2 from both sides.

The inverse function of \( y = \sin(\sqrt{x + 2}) \) is \( y = (arcsin(x))^2 - 2 \).

Find the inverse function of \( y = e^{\sin(\sqrt{x} + 2)} \)

**Solutions.** We set

\[
x = e^{\sin(\sqrt{T} + 2)}
\]

take the natural log of both sides:

\[
\ln(x) = \sin(\sqrt{y} + 2),
\]

then take the arcsine of both sides

\[
\arcsin(\ln(x)) = \sqrt{y} + 2,
\]

and then subtract 2 from both sides

\[
\arcsin(\ln(x)) - 2 = \sqrt{y},
\]

and finally square both sides.

The inverse function of \( y = e^{\sin(\sqrt{T} + 2)} \) is \( y = (arcsin(\ln(x) - 2))^2 \).

Find the inverse function of \( y = \sin(\arccos x) \)

**Solutions.** First we simplify \( \sin(\arccos x) \). Draw a right triangle with a hypotenuse of length 1 and an acute angle \( \theta \) with adjacent side of length \( x \). The side opposite of \( \theta \) has length (by the Pythagorean theorem) \( \sqrt{1 - x^2} \). So the cosine of \( \theta \) is just \( \sqrt{1 - x^2} \).

We have simplified \( y = \sin(\arccos x) \) to \( y = \sqrt{1 - x^2} \). It happens that the inverse function of \( y = \sqrt{1 - x^2} \) obeys the equation

\[
x = \sqrt{1 - y^2} \text{ so } x^2 = 1 - y^2 \text{ so } y^2 = 1 - x^2 \text{ so } y = \sqrt{1 - x^2}.
\]

(That is \( y = \sqrt{1 - x^2} \) is its own inverse function!)
REMEMBER: When faced with an inverse trig function, think about the triangle the function creates!

In the next presentation, we will look at trig identities and equations.

(End)