Applications of exponential functions abound throughout the sciences. Exponential functions are the primary functions that scientists work with. Here are some examples.

**Exponential growth.**
For most biological systems, the amount of growth in the population is directly proportional to the size of the population. (The more adult animals there are, the more mating pairs there are and so the more newborn animals there will be!) For this reason, biological populations can be modeled by exponential growth.

Similarly, investment strategies can often be modeled by exponential growth since the more money one has, the more one is likely to earn in by investing that money.

A typical exponential growth function has the form

\[ P(t) = P_0 e^{kt} \]

where \( t \) is the independent variable (usually standing for time) and \( P_0 \) and \( k \) are constants that come with the population model. \( P_0 \) will typically be the “initial population”; it is, after all, equal to \( P(0) \) since \( e^0 = 1 \).

**Example.** Suppose the population of gray wolves in Yellowstone is approximated by \( P(t) = P_0 e^{0.3t} \) where \( t \) is measured in years after the first mating pair were re-introduced in 1995. What is the population of wolves in 1995 (at \( t = 0 \) years), in 1998 (\( t = 3 \) years), in 2010 (\( t = 15 \) years), and in 2025 (\( t = 30 \) years)?

**Solutions.** Here \( P_0 = 2 \) since this the population at the beginning of the experiment, when \( t = 0 \) in 1995. So obviously \( P(0) = 2 \).

Three years later the population according to this model is \( P(3) = 2e^{0.9} \approx 4.919 \).

In 2025, according to this model, \( P(30) = 2e^9 \approx 16206 \).

Obviously a purely exponential model of biological growth is simplistic. It does not take into account death (wolves don’t live 30 years) nor does it take into account limits on space and resources (the Yellowstone environment can probably not maintain much more than 200 wolves.)

One might also note that when wolves were re-introduced in Yellowstone in 1995, it was not just a single mating pair that was introduced, but several packs.
Mathematics in biology (logistic growth)

Populations grow exponentially until they meet some type of limiting factor such as food supply or space limitations. At that point there needs to be additional mathematical formulas to take into account the limiting factors.

For example, the Gomperz function \( f(t) = ae^{be^{ct}} \) models population growth in confined spaces.

Sometimes (as in the wolves at Yellowstone) the growth begins to flatten out towards a particular “ceiling” given by the logistics curve (or "Verhulst curve".)

The logistics curve is an example of a sigmoid or “S-shaped” curve. The standard logistics curve is the graph of the function \( f(x) = \frac{1}{1+e^{-x}} \).

Exponential decay

Another application of exponential functions is exponential decay.

If \( k \) is positive, the graph of \( g(x) = e^{kx} \) has the familiar exponential function explosion seen in the earlier graph of \( f(x) = 2^x \). (Indeed, if \( k \approx 0.693 \), the curves \( y = 2^x \) and \( y = e^{kx} \) are the same.)

But what if \( k \) is negative? The graph of \( g(x) = e^{-x} \) is reflected about the \( y \)-axis, so the curve rises dramatically to the left and falls towards zero on the right.

This is exponential decay. It is modeled by population decline. For example, one might be attempting to eradicate an infectious disease like polio, and, over time, model the decrease in polio cases by a decaying exponential function.

One form of exponential form is radioactive decay. We will look at radioactive decay in a later lesson.
Newton’s Law of Cooling

Another form of exponential decay occurs in Newton’s Law of Cooling. One can model the cooling of a hot liquid in the open air by comparing the difference in current temperature to air temperature \((T(t) - T_a)\) with the initial difference in temperature \((T(0) - T_a)\).

The ratio of these two will generally decay exponentially so that
\[
\frac{T(t) - T_a}{T_0 - T_a} = e^{-kt}
\]

Here \(k\) is a constant that depends on the liquid and the environment. It is common to clear denominators and solve for \(T(t)\) so that \(T(t) = T_a + (T(0) - T_a)e^{-kt}\). We might also (as in the population models) write \(T_0\) for the initial temperature \(T(0)\) and therefore express Newton’s Law of Cooling as
\[
T(t) = T_a + (T_0 - T_a)e^{-kt}
\]

Along with modeling the cooling of a hot liquid, Newton’s Law of Cooling can be used as a first approximation in modeling the cooling of something more complicated, such as the temperature of a corpse (in forensic chemistry.)

Exponential Functions

In the next presentation we will look at another applications of exponential functions, compound interest.

(End)
Suppose we wish to invest $200 at 5% annual interest. After one year we have earned interest of $10 = (200)(0.05) and so our investment is $200 + $10 = $210. If we invest that money again (all of it at 5%), then in the next year we earn interest of $10.50 = (210)(0.05) and so our investment has grown to $210 + $10.50 = $220.50.

After each interest period, the (future) value $A$ of our investment is

$$A = P(1 + r)$$

where $P$ is the amount invested and $r$ is the interest rate.

If we continue to invest our money compounded across $n$ investment periods, the formula becomes

$$A = P(1 + r)(1 + r) \cdots (1 + r) = P(1 + r)^n$$

Notice here that $r$ is the interest rate across the compound period (not necessarily the annual rate!) and $n$ is the number of compound periods. This is an easy and natural formula for computing compound interest. There is no need to work with more complicated formulas!

### Some worked problems.

For example, in order to buy a car, Leticia borrows $5000 at 6% annual interest, compounded annually. How much does she owe after five years?

**Solution.**

The interest period is one year since the investment compounds only annually. The number of interest periods is 5. Therefore the future value of the loan is

$$5000(1 + .06)^5 = 5000(1.33823) = \, \boxed{6691.13}.$$

Suppose instead that Leticia borrows $5000 at 6% annual interest, compounded monthly. How much does she owe after five years?

**Solution.**

The interest period is one month since the investment compounds every month. The interest rate per month is $0.06/12 = 0.005$. The number of interest periods is 60 (5 years of 12 months.) Therefore the future value of the loan is

$$5000(1 + .06/12)^{60} = 5000(1.005)^{60} = 5000(1.348850) = \boxed{6774.25}.$$

Notice that this second loan has a greater value ($6774.25 versus $6691.13) since the interest is compounding more frequently.

### Predatory lending practices

Compound interest can grow dramatically. If one does not understand this, one will fall victim to predatory loans. Here is an example.

**Now PayDay Loans** offers loans to cover you to your next payday, generally assumed to be two weeks away. In Texas, Now Payday Loans charges approximately $25 for each $100 borrowed if you borrow money for 15 days.

Suppose that you borrow $100 from a Payday loan organization with a 25% interest rate compounded every 15 days. But after 15 days, you pay nothing back, and so must pay interest on interest, so that your interest compounds. How much will your loan cost you if pay it all back after

1. One month? (Assume a month is 30 days.)
2. Two months?
3. One year? (Assume a year is 360 days.)

**Solution.**

1. $100 \times (1.25)^2 = 156.25.$
2. $100 \times (1.25)^4 = 241.14.$
3. $100 \times (1.25)^{24} = 21175.82.$
Payday loans are indeed predatory. Here is a paragraph from from a New York Times article on their practices.

“While the loans are simple to obtain – some online lenders promise approval in minutes with no credit check – they are tough to get rid of. Customers who want to repay their loan in full typically must contact the online lender at least three days before the next withdrawal. Otherwise, the lender automatically renews the loans at least monthly and withdraws only the interest owed. Under federal law, customers are allowed to stop authorized withdrawals from their account. Still, some borrowers say their banks do not heed requests to stop the loans.”

Credit card, debit card and other “cash advance” companies prey on the mathematically ignorant. It is very important, when one borrows, to pay attention to the interest rates, to pay attention to the true interest rate, taking into account the fees!

Most organizations in the US are now required by law to list (in small print!) the APR, the annual percentage rate. In some cases the APR is over 1300 percent!

Here is part of the (required) APR report for Now PayDay loans – notice the column under APR – see how high the true interests rates are!

<table>
<thead>
<tr>
<th>APR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1310.05%</td>
</tr>
<tr>
<td>1153.10%</td>
</tr>
<tr>
<td>1028.30%</td>
</tr>
<tr>
<td>924.96%</td>
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<td>842.04%</td>
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<td>772.91%</td>
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<td>447.01%</td>
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<tr>
<td>409.23%</td>
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<tr>
<td>392.70%</td>
</tr>
<tr>
<td>377.50%</td>
</tr>
<tr>
<td>353.46%</td>
</tr>
</tbody>
</table>

If one pays attention to interest rates and is willing to do a little exponential mathematics, one can see that most short term loans should be avoided.
A second formula for compound interest

The formula \( A = P(1 + r)^n \) for computing the future value of a compound interest investment is easy to derive and easy to use. However, some textbooks will add a second formula for students to memorize. Personally, I don’t use this second formula... but I will describe it here.

Suppose that one is investing interest at an annual interest rate \( r \) compounded \( n \) times a year for \( y \) years. Then our equation for compound interest tells us that since the interest rate per compound period is now \( r/n \) and since the number of compound periods is \( ny \) then

\[
A = P(1 + \frac{r}{n})^{ny}
\]

In this case, \( r \) is not the interest rate per compound period but is, instead, the annual interest rate.

Let’s explore how one might use this new equation by working one problem with the two different formulas.

A worked example. I wish to borrow \$1000\$ at 3% annual interest. The loan will compound monthly. How much do I owe after five years?

**Solution #1.** Note that the compound period is a month (one-twelfth of a year) and so the interest rate per compound period is \( r = 3%/12 = 0.25\% = 0.0025 \). The number of compound periods is \( n = 60 \), since there are 60 months in 5 years.

\[
A = 1000(1 + 0.0025)^{60} \approx \$1,161.62
\]

**Solution #2.** Using the equation on the previous slide, writing \( r = 3\% = 0.03 \) as the annual interest rate and \( y = 5 \) as the number of years. Since the compound period is a month, then \( n = 12 \) in that equation and so we see that

\[
A = 1000(1 + \frac{0.03}{12})^{12 \cdot 5} \approx \$1,161.62
\]

It does not matter whether one always uses the equation I suggest (in Solution #1)

\[
A = P(1 + r)^n \tag{1}
\]

or whether one use the new equation often given in textbooks or by tutors,

\[
A = P(1 + \frac{r}{n})^{ny} \tag{2}
\]

The two equations differ only in the meaning of \( r \) and \( n \).

If compounding interest makes an investment or loan grow rapidly, then “compounding continuously” should give the most rapid rise. When one compounds continuously, the interest is viewed as drawing interest on it, as soon as it occurs. This means that one views the process as a limit in which the interest period shrinks to daily, then hourly, then second by second and eventually shrinks to zero.

If interest is compounded continuously at an annual rate of \( r \), then the future value of the investment (or loan) after one year is \( A = e^r \). The future value of the investment after \( t \) years is

\[
A = e^{rt}
\]

Notice how the “natural” base \( e \) has entered our picture!
Some worked problems.

Suppose that you borrow $1000 at annual interest rate \( i \), compound continuously. For each interest rate \( i \), and length of time \( t \), compute the future value of the loan. (Note: 3\% might be a typical bank loan, 20\% a credit card loan, 600\% the loan you would receive from Now PayDay.)

1. \( i = 3\% \), \( t = 1 \) year.
2. \( i = 20\% \), \( t = 1 \) year.
3. \( i = 600\% \), \( t = 1 \) year.
4. \( i = 3\% \), \( t = 20 \) years.
5. \( i = 20\% \), \( t = 20 \) years.
6. \( i = 600\% \), \( t = 20 \) years.

Solutions.

1. \( e^{it} = e^{0.03} \approx 1.0304545 \). Multiply this by $1000 to get $1030.45
2. \( e^{it} = e^{0.20} \approx 1.221403 \). Multiply this by $1000 to get $1221.40
3. \( e^{it} = e^{6.00} \approx 403.42879 \). Multiply this by $1000 to get $403,428.79
4. \( e^{it} = e^{0.60} \approx 1.8221188 \). Multiply this by $1000 to get $1822.12
5. \( e^{it} = e^{4} \approx 54.5981500 \). Multiply this by $1000 to get $54,598.15

In the next presentation we will look at inverse functions of exponential functions, that is, logarithms.

(End)