1.2 Graphs of functions

1.2.1 Using ordered pairs to draw functions

If we describe our function using an equation \( y = f(x) \) with inputs \( x \) and outputs \( y \), then we may view the inputs, \( x \), as elements of a horizontal line in the plane and record outputs \( y \) on a vertical line. The graph of a function in the Cartesian plane is the set of values \( (x, f(x)) \).

Combining functions with the geometry of the plane gives us a nice visual way to see and understand a function. This idea was first introduced in the 17th century by (among others) Rene Descartes and so the plane in which we draw our graph is called the Cartesian plane.

We graph the function \( f \) with \( x \) values increasing along a horizontal axis and \( y \) values increasing along a vertical axis. It is customary to draw the \( x \)-axis along the horizontal line where \( y = 0 \) and draw the \( y \)-axis along the vertical line where \( x = 0 \).

Most graphs of functions \( y = f(x) \) can be sketched by creating a table of values \( (x, y) \) and then making some reasonable assumptions as to how these points should be connected. For example, consider the function \( f(x) = x^2 \). We can create a table of values. (Notice that we input some values which are not integers!)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.25</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.25</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

We can then plot the points \((-2, 4), (-1, 1), (-0.5, 0.25), ...\) on the Cartesian plane and use these points to guide us on filling in the rest of the curve. Of course modern software (and graphing calculators) can do this for us, plotting hundreds of points and connecting the dots....

**Figure 5.** The graph of the function \( f(x) = x^2 \) (created by the author, using Sage.)
Some worked examples.
For each function, create a table of values (with at least 5 points, where at least one of which does not have integer value for $x$) and then graph the function.

1. $f(x) = x^3 - x$
2. $f(x) = |x|$
3. $f(x) = \sqrt{x}$

Solutions. (The graphs in this section were generated by the author using Sage.)

1. Here is a table of a few values for the function $f(x) = x^3 - x$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>-0.5</td>
<td>-0.625</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

If we connect the dots, we should get something like this:

2. Here is a table of values for the function $f(x) = |x|$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

If we connect the dots, we should get something like this:
3. Here is a table of values for the function $f(x) = 3\sqrt{x}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = 3\sqrt{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-8</td>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>-\frac{1}{8}</td>
<td>-\frac{1}{2}</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
</tr>
</tbody>
</table>

If we connect the dots, we should get something like this:

1.2.2 Intercepts of the graph of a function

An $x$-intercept is a place where the graph touches the $x$-axis, that is, where $y = 0$. A $y$-intercept is a place where $x = 0$ so that the function has a point on the $y$-axis. Given an equation for a function $f(x)$, it is easy to determine the $y$-intercept: simply compute $f(0)$.

It can be harder to find the $x$-intercepts of the function since we are seeking to solve the equation $f(x) = 0$ for $x$-values. Here are some examples of how one might approach this problem.
Examples.

Find the intercepts of the following functions

1. \( f(x) = x^2 - 2x - 3 \).
2. \( g(x) = \frac{(x + 1)(x^2 - 6x + 8)}{x + 2} \).
3. \( h(x) = \frac{1}{x + 2} + \frac{2x - 3}{2x + 1} + x - 5 \).

Solutions.

1. \( f(x) = x^2 - 2x - 3 \) has \( y \)-intercept \((0, -3)\) since \( f(0) = -3 \). To find the \( x \)-intercepts of the function, we need to solve the equation \( 0 = x^2 - 2x - 3 \). We may factor \( x^2 - 2x - 3 = (x - 3)(x + 1) \) and solve \( 0 = (x - 3)(x + 1) \). The product of two expressions is zero if and only if one of the expressions is zero so either \( 0 = x - 3 \) or \( 0 = x + 1 \). Thus the \( x \)-intercepts occur where \( x = 3 \) or \( x = -1 \). So the \( x \)-intercepts are \((-1, 0)\) and \((3, 0)\).

2. The function \( g(x) \) is a “rational function”, that is, a ratio of two polynomials. We should take a moment and consider the denominator of this function. The denominator is zero when \( x = -2 \) and so the function \( g(x) \) is undefined at \( x = -2 \). \( x = -2 \) cannot then give us any point on the graph, much less an intercept.

To find the \( y \)-intercept of \( g(x) = \frac{(x + 1)(x^2 - 6x + 8)}{x + 2} \), merely plug in 0: \( g(0) = \frac{(1)(8)}{2} = 4 \) so the \( y \)-intercept is \((0, 4)\).

To find the \( x \)-intercept of \( g(x) = \frac{(x + 1)(x^2 - 6x + 8)}{x + 2} \), we set the function equal to zero and solve:

\[
0 = \frac{(x + 1)(x^2 - 6x + 8)}{x + 2}.
\]

Multiply both sides by the numerator \( x + 2 \)
\[
0 = (x + 1)(x^2 - 6x + 8)
\]

and factor the expression on the right:
\[
0 = (x + 1)(x - 2)(x - 4)
\]

The expression on the right is zero whenever any of its terms are zero, so the \( x \)-intercepts are \((-1, 0)\), \((2, 0)\) and \((4, 0)\).

3. The \( y \) intercepts of the function \( h(x) = \frac{1}{x+2} + \frac{2x-3}{2x+1} + x - 5 \) occur where \( x = 0 \) so

\[
y = \frac{1}{0 + 2} + \frac{0 - 3}{0 + 1} + 0 - 5 = \frac{1}{2} - 3 - 5 = -\frac{15}{2}.
\]

Thus the \( y \)-intercept is \((0, -\frac{15}{2})\).

The \( x \)-intercept is where \( y = 0 \) so we examine the equation

\[
0 = \frac{1}{x + 2} + \frac{2x - 3}{2x + 1} + x - 5
\]

and solve for \( x \).
Multiply both sides by the denominators $x + 2$ and $2x + 1$ to clear denominators and we have

$$0 = (2x + 1) + (x + 2)(2x - 3) + (x + 2)(2x + 1)(x - 5).$$

Expanding this out and simplifying, we have

$$0 = (2x + 1) + (2x^2 + x - 6) + (2x^3 - 5x^2 - 23x - 10)$$

so

$$0 = 2x^3 - 3x^2 - 20x - 15.$$

A graphing calculator helps us find that $x = -1$ is one of the solutions to this equation; there are two more solutions which are a bit more complicated, involving the quadratic formula; these are $\frac{1}{4}(5 \pm 3\sqrt{5})$. We will look more at this problem later.

### 1.2.3 Intervals in which the function rises or falls

A variety of applications in science require that we understand the intervals where a function is rising or falling and that we be able to compute the local minimums and locals maximums of a function. We will want to know the $x$ values for which $f(x)$ is as large as possible (in some specific region) or where $f(x)$ is as small as possible. Later, in calculus, we will find a universal approach to this very important problem, but at this stage, we are content to draw graphs and find minimums and maximums visually.

**Example.**

Consider the function $f(x) = x^4 - 8x^2$ with graph given below:

![Graph of $f(x) = x^4 - 8x^2$](image)

*Figure 6. The graph of the quartic polynomial $f(x) = x^4 - 8x^2$.***

Where is this function increasing? Where is it decreasing? What are the low points on the curve (local minimums)? What are the high points on the curve (local maximums)?

From the picture we can see that the function drops to the point $(-2, -16)$, rises to the origin $(0, 0)$, drops again to the point $(2, -16)$ and then rises after that. So the function is decreasing for $x$-values in the region $(-\infty, -2) \cup (0, 2)$, and rising in the region $(-2, 0) \cup (2, \infty)$.

The local minimums are $(-2, -16)$ and $(2, -16)$; a local maximum occurs at the point $(0, 0)$. 
### 1.2.4 The vertical line test definition of a function

In the definition of function, we require that a function have a unique output for each input. The geometric version of this requirement is the “vertical line test”. The points in the plane corresponding to a fixed $x$-value form a vertical line. For example, the line $x = 3$ is a vertical line consisting of all points of the form $(3, y)$ for any value of $y$.

If our graph is the graph of a function then each vertical line will touch the graph in exactly one point, at the $y$ value $y = f(x)$. So to see if our graph is the graph of a function, draw vertical lines and see if these lines intersect the graph exactly once.

In the image below (figure 7), a vertical line (in yellow) corresponding to a particular $x$-value hits the curve at exactly one point. This should be true for all the elements in the domain of the function.

![Figure 7. The vertical line test](from Wikipedia, author Wybailey, available under the Creative Commons license.)

### 1.2.5 Piecewise functions

Functions need not be described by an equation. Some, like the function “color” described in section 1.1, will not involve equations at all.

Other functions can be described, not by a single formula, but by a collection of them. Many functions involve jumps of some type, or different formulas depending upon various subsets of the domain. A piecewise function is a function which is described for “pieces” of the real line. It will obey one rule in one region and another rule in another region.

One of the simplest such functions might be described by the question, “How much does it cost to mail a letter?” Let us assume that we are mailing a letter that weights less than 4 ounces. The US Postal Service offers a price chart which is copied below.
The leftmost column describes an input variable, the weight of the letter in ounces. The second column gives the cost if the first-class mail is an ordinary letter. One should interpret this information as follows:

- If the letter is between 0 and 1 ounce in weight, the cost is 45 cents.
- If the letter is more than 1 ounce in weight, but no more than 2 ounces, the cost is 65 cents.
- If the letter is more than 2 ounces in weight, but no more than 3 ounces, the cost is 85 cents.
- If the letter is more than 3 ounces in weight, but no more than 3.5 ounces, the cost is $1.05 (105 cents)
- If the letter is more than 3.5 ounces in weight (but less than 4 ounces), then one should use a large envelope and the cost is $1.50.

(Although the USPS chart described mail costs for mail up to 13 ounces, for convenience, I have assumed that we are not mailing anything heavier than 4 ounces.)

We can describe this “cost of first-class” function as follows. Here the input $x$, is weight in ounces, and the output $f(x)$ is cost in cents. The cost of a letter of weight $x$ is then $f(x) = \begin{cases} 45, & \text{if } 0 \leq x \leq 1 \\ 65, & \text{if } 1 < x \leq 2 \\ 85, & \text{if } 2 < x \leq 3 \\ 105, & \text{if } 3 < x \leq 3.5 \\ 150, & \text{if } 3.5 < x \leq 4 \end{cases}$

Below is the graph of this function. Note the use of open circles to describe points not part of the function graph and closed dark circles to describe points that are on the function graph, at the end of a curve.
Three worked examples.

Consider the function defined in “pieces” as follows:

\[ f(x) = \begin{cases} 
2x + 1, & \text{if } x \leq 0 \\
x^2 + 1, & \text{if } x > 0
\end{cases} \]

Compute \( f(2) \), \( f(0) \), \( f(-2) \) and graph this function.

Solution.

Here, if \( x \leq 0 \) then the function obeys the rule \( f(x) = 2x + 1 \). Thus \( f(-2) = 2(-2) + 1 = -3 \) and \( f(0) = 2(0) + 1 = 1 \). But if \( x > 0 \) then \( f(x) = x^2 + 1 \) and so \( f(2) = 2^2 + 1 = 5 \). The graph of the function appears in figure 10. Notice how the two “pieces” are glued together; we can see the line \( y = 2x + 1 \) in the region where \( x \leq 0 \) and the parabola \( y = x^2 + 1 \) in the region to the right of the \( y \)-axis.
Example 2. Another example: graph the function

\[ f(x) = \begin{cases} 
2x + 2, & \text{if } x \leq 1 \\
4, & \text{if } x > 1 
\end{cases} \]

Then compute \( f(0), f(1), f(2) \) and \( f(100) \).

Solution. Since \( x = 0 \) and \( x = 1 \) both fall in the region where \( x \leq 1 \), we have that \( f(0) = 2(0) + 2 = 2 \) and \( f(1) = 2(1) + 2 = 4 \). Since \( x = 2 \) and \( x = 100 \) both fall in the region where \( x > 1 \) then \( f(2) = 4 \) and \( f(100) = 4 \).

Here is the graph:

![Graph of Example 2](image1)

Figure 11. A function defined in two pieces

Example 3. Graph the function below and compute \( f(-1), f(0.5), f(1.5), f(2.5), f(3.5) \).

\[ f(x) = \begin{cases} 
x^2, & \text{if } x \leq 0 \\
0, & \text{if } 0 \leq x < 1 \\
1, & \text{if } 1 \leq x < 2 \\
2, & \text{if } 2 \leq x < 3 \\
x, & \text{if } x \geq 3 
\end{cases} \]

Solution. \( f(-1) = (-1)^2 = 1 \) since \( x = -1 \leq 0 \). But \( f(-0.5) = 0 \) since \( 0 \leq 0.5 < 1 \) Similarly \( f(1.5) = 1, f(2.5) = 2 \) and \( f(3.5) = 3.5 \). The graph is:

![Graph of Example 3](image2)

Figure 12. A function defined in five pieces
Note in Figure 12 the use of closed dark circle to indicate that the point (such as \((1, 1)\)) is on the graph and the use of light, open circles to indicate that a point (such as \((1, 0)\) or \((2, 1)\)) is not on the graph.

### 1.2.6 Other resources for graphing functions

In the free textbook, *Precalculus*, by Stitz and Zeager (version 3, July 2011, available at [stitz-zeager.com](http://stitz-zeager.com)) this material is covered in section 1.6.

In the free textbook, *Precalculus, An Investigation of Functions*, by Lippman and Rasmussen (Edition 1.3, available at [www.opentextbookstore.com](http://www.opentextbookstore.com)) this material is covered in section 1.3.

In the textbook by Ratti & McWaters, *Precalculus, A Unit Circle Approach*, 2nd ed., c. 2014, [here at Amazon.com](http://www.amazon.com) this material appears in section 1.4. In the textbook by Stewart, *Precalculus, Mathematics for Calculus*, 6th ed., c. 2012, [here at Amazon.com](http://www.amazon.com) this material appears in sections 2.2 and 2.3. (In July 2013 the first textbook was $147 at Amazon.com and the second textbook was $136 at Amazon.com They are even more expensive in campus bookstores.)

There are lots of online resources for learning about graph of elementary functions. Here are some I recommend.

1. [Dr. Paul’s online notes on graphing functions](http://www.math.colostate.edu/~bryantma PattyF.pdf)
2. [A Khan Academy video on graphing functions](https://www.khanacademy.org)

**Homework.**

As class homework, please complete **Worksheet 1.2, Functions and their graphs**, available through the class webpage.