3.3 Convergence Tests for Infinite Series

3.3.1 The integral test

We may plot the sequence \( a_n \) in the Cartesian plane, with independent variable \( n \) and dependent variable \( a \).

The sum \( \sum_{n=1}^{n} a_n \) can then be represented geometrically as the area of a collection of rectangles with height \( a_n \) and width 1. This geometric viewpoint suggests that we compare this sum to an integral. If \( a_n \) can be represented as a continuous function of \( n \), for real numbers \( n \), not just integers, and if the sequence \( a_n \) is decreasing, then \( \sum_{n=1}^{m} a_n \) looks a bit like area under the curve \( a = a(n) \).

In particular,

\[
\sum_{n=1}^{m} a_n > \int_{n=1}^{n+1} a_n \, dn > \sum_{n=2}^{m+2} a_n
\]

For example, let us examine the first 10 terms of the harmonic series

\[
\sum_{n=1}^{10} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}.
\]

If we draw the curve \( y = \frac{1}{x} \) (or \( a = \frac{1}{n} \)) we see that

\[
\sum_{n=1}^{10} \frac{1}{n} > \int_{1}^{11} \frac{1}{x} \, dx > \sum_{n=2}^{11} \frac{1}{n} = \sum_{n=1}^{10} \frac{1}{n} - 1 + \frac{1}{11}.
\]

(See Figure 1, copied from Wikipedia)

Now \( \int_{1}^{11} \frac{1}{x} \, dx = \ln(11) - \ln(1) = \ln(11) \) so

\[
\sum_{n=1}^{10} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} > \ln(11)
\]

and

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} < \ln(11) + (1 - \frac{1}{11}).
\]

So we may bound our series, above and below, with some version of the integral \( \int \frac{1}{x} \, dx \).

If we allow the sum to turn into an infinite series, we turn the integral into an improper integral. The integral \( \int_{1}^{\infty} \frac{1}{x} \, dx \) diverges to infinity, so the sum \( \sum_{n=1}^{\infty} \frac{1}{n} \) must also diverge.

In general, the convergence or divergence of a series is equivalent to convergence or divergence of the associated improper integral.

**Theorem.** (Integral test)

Suppose \( a(n) \) is a continuous function of \( n \) for all positive real numbers \( n \). Then the series \( \sum_{n=1}^{\infty} a_n \) converges or diverges exactly when \( \int_{1}^{\infty} a(x) \, dx \) converges or diverges.
Warning! If the integral converges then the associated series converges. But they might not converge to the same thing! For example, the integral \( \int_1^\infty \frac{1}{x^2} \, dx \) converges to 1. But the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges to a number larger than 1! (Indeed, Euler discovered around 1735 that the series converges to \( \frac{\pi^2}{6} \).)


### 3.3.2 \( p \)-series

For what values of the exponent \( p \) does the series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converge?

According to the integral test, we can ask, instead, “for what values of the exponent \( p \) does the integral \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converge?"

The improper integral is evaluated as follows:

1. If \( p > 1 \) then
   \[
   \int_{1}^{\infty} \frac{dx}{x^p} = \lim_{b \to \infty} \int_{1}^{b} x^{-p} \, dx = \lim_{b \to \infty} \left[ \frac{1}{1-p} x^{-p+1} \right]_{1}^{b} = \lim_{b \to \infty} \frac{1}{1-p} (b^{-p+1} - 1)
   \]

   Since \( p > 1 \) then \( \lim_{b \to \infty} b^{-p+1} = 0 \) and so this integral and the associated series converge.

2. If \( p = 1 \) then
   \[
   \int_{1}^{\infty} \frac{1}{x^p} = \int_{1}^{\infty} \frac{dx}{x} = \lim_{b \to \infty} \int_{1}^{b} \frac{dx}{x} = \lim_{b \to \infty} \ln x \bigg|_{1}^{b} = \lim_{b \to \infty} \ln b = \infty.
   \]

   So the integral and the associated series diverge.

   But then the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is the harmonic series. We already knew that it diverges!
3. If \( p < 1 \) then
\[
\int_1^\infty \frac{dx}{x^p} = \lim_{b \to \infty} \frac{1}{1-p} (b^{-p+1} - 1)
\]

Since \( p > 1 \) then \( \lim_{b \to \infty} b^{-p+1} = \infty \) and so this integral and the associated series converge.
So the integral and the associated series diverge.

**Conclusion.** (\( p \)-test)

The series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges if and only if \( p > 1 \).

### 3.3.3 Comparison tests

The integral test provides information about convergence and divergence of a series, if we can evaluate the integral. Even if we can evaluate the integral and the series converges, all the test tells us is that the series converges. It does *not* tell us the limit of the series.

We now focus on convergence/divergence and give up attempting to determine the exact limit of a converging series. Once we are willing to do this, it turns out that there are a number of techniques that will provide us with convergence/divergence information.

The first test is the **comparison test**. Think of a converging series as one which is “small”, that is, the terms go to zero so rapidly that the infinite sum converges to a finite real number. If another series is “smaller” than a converging series then it too should converge.

On the other hand, a series which diverges to infinity is “large” – that is, the terms, even if they go to zero, provide a sum which grows beyond bounds. Therefore another series which is term-by-term larger than a diverging series, should also diverge to infinity.

We can make this precise.

If \( a_n \geq b_n \geq 0 \) for all \( n \) and \( \sum_{n=0}^{\infty} b_n \) diverges then \( \sum_{n=0}^{\infty} a_n \) diverges.

If \( 0 \leq a_n \leq b_n \) for all \( n \) and \( \sum_{n=0}^{\infty} b_n \) converges then \( \sum_{n=0}^{\infty} a_n \) converges.

In some cases, we can’t do a strict comparison test between two series but the series begin, in the limit, to “look alike.”

This leads us to the **limit comparison test**:  

If \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \) then \( \sum a_n \) and \( \sum b_n \) either both converge or both diverge.

Let’s make this clear with some examples.

1. We know, via the integral test, that the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges. Since \( \frac{n}{n^2 - 0.5} > \frac{n}{n^2} = \frac{1}{n} \)  
then the series \( \sum_{n=1}^{\infty} \frac{n}{n^2 - 0.5} \) diverges by the (strict) comparison test.

2. What about \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \)? The straightforward comparison test does not help us here since \( \frac{n}{n^2 + 1} < \frac{1}{n} \)  
and it does us no good to examine a series which is term-by-term *smaller* than one which diverges.
However, if we compare the two in the limit, and compute the ratio \( \frac{n}{n^2 + 1} / \frac{1}{n} = \frac{n^2}{n^2 + 1} \) then, in the limit, as \( n \to \infty \), we have \( \frac{n^2}{n^2 + 1} \to 1 \). In the limit, the two series become indistinguishable and so the limit comparison test assures us that as far as convergence/divergence is concerned, the two series act the same. They both diverge.

3. We know, via the integral test, that the series \( \sum_{n=1}^{\infty} 1/n^2 \) converges.

What about

\[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \]

or

\[ \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} \]

In the first case, we can use the strict comparison test and note that since \( \frac{1}{n+1} < \frac{1}{n} \) then \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \) is smaller, term-by-term, than a converging series and so it converges.

In the second case, we know that \( \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} \) is term-by-term larger than \( \sum_{n=2}^{\infty} \frac{1}{n^2} \) and this is not helpful. But the limit comparison test is helpful since the limit (as \( n \to \infty \)) of \( \left( \frac{1}{n^2 + 1} \right) / \left( \frac{1}{n^2} \right) = \frac{n^2}{n^2 + 1} \) is 1. Since \( \sum_{n=1}^{\infty} 1/n^2 \) converges then so does \( \sum_{n=2}^{\infty} 1/n^2 \) and since \( \sum_{n=2}^{\infty} 1/n^2 \) converges then \( \sum_{n=2}^{\infty} 1/n^2 - 1 \) converges.

The comparison tests are valuable when the series we are examining is “close” to a series we already understand. Often a little algebra allows us to appropriately compare the series in front of us with the series we know and get the same result. (But we must make sure the comparison is legitimate! For example, it does no good to compare a series with a diverging one if our series is term-by-term smaller!)

3.3.4 Big-oh and little-oh

(Under construction. See [http://en.wikipedia.org/wiki/Big_O_notation](http://en.wikipedia.org/wiki/Big_O_notation) and notes stored with another course....)

3.3.5 Alternating Series

We know that if a series converges, the terms must go to zero. But there are series (such as the harmonic series) where the terms go to zero but the series diverges. So if \( a_n \to 0 \) then we know nothing about \( \sum a_n \)!

Does the terms going to zero ever really help us? There is one case where we \( a_n \to 0 \) is truly helpful.

The alternating series test

If \( \sum_{n=0}^{\infty} a_n \) is an alternating series and the terms \( a_n \) go to zero then the series converges.

Absolute convergence.

A series \( \sum_{n=0}^{\infty} a_n \) converges absolutely if the series \( \sum_{n=0}^{\infty} |a_n| \) converges.
If a series $\sum_{n=0}^{\infty} |a_n|$ converges then so does $\sum_{n=0}^{\infty} a_n$.

An alternating series $\sum a_n$ converges conditionally if the series converges but $\sum |a_n|$ does not; in this case, the alternating terms are critical to the convergence of the series.

### 3.3.6 Ratio and root tests

A geometric series is detectable by the fact that the ratio $\frac{a_{n+1}}{a_n} = r$ is a constant, independent of $n$.

It turns out that many series which are not geometric have a “ratio” in the limit. Guided by the convergence tests for a geometric series, we have the following.

**The ratio test** (important!)

If the limit $\rho = \lim_{n \to \infty} |\frac{a_{n+1}}{a_n}|$ exists and if less than one, the series converges.

If the limit $\rho$ exists and is greater than one, the series diverges.

Note. Unlike the geometric series, it turns out that we know nothing here if the ratio is exactly 1. The series may or may not converge.

(Some examples: the harmonic series, the alternating harmonic series, the Basel series.)

We will use the ratio test almost exclusively when we get to power series and Taylor series.

**The root test**

Another version of the question “does it eventually look geometric?” leads us to the root test.

If the limit $\rho = \lim_{n \to \infty} \sqrt[n]{a_n}$ exists and if less than one, the series converges.

If the limit $\rho$ exists and is greater than one, the series diverges.

### 3.3.7 Summary of convergence tests

Here are a few tests we can use for convergence of a series $\sum_{n=0}^{\infty} a_n$.

(See the flowchart which accompanies this.)

1. A series to recognize: Geometric!
   The ratio $\frac{a_{n+1}}{a_n}$ is a constant, independent of $n$.

2. The “terms-must-go-to-zero” test for divergence.
   If a series converges then the terms $a_n$ must converge to zero. So if the terms $a_n$ do not converge to zero then the series must diverge.

3. The integral test.
   If $a_n$ is an integrable function of $n$ then the series $\sum_{n=0}^{\infty} a_n$ converges if and only if $\int_{x=0}^{\infty} a(x) \, dx$ does.

(3a) $p$-series (and harmonic series.)
\[ \sum \frac{1}{n^p} \text{ converges if and only if } p > 1. \]

The harmonic series is the series where \( p = 1 \); it diverges.

4. Be alert for the partial fractions decomposition and possibly a telescoping series.

5. The comparison test.
   
   If \( a_n \geq b_n \geq 0 \) for all \( n \) and \( \sum_{n=0}^{\infty} b_n \) diverges then \( \sum_{n=0}^{\infty} a_n \) diverges.
   
   If \( 0 \leq a_n \leq b_n \) for all \( n \) and \( \sum_{n=0}^{\infty} b_n \) converges then \( \sum_{n=0}^{\infty} a_n \) converges.

6. The limit comparison test
   
   If \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \) then \( \sum a_n \) and \( \sum b_n \) both converge or diverge.

7. The alternating series test
   
   If \( \sum_{n=0}^{\infty} a_n \) is an alternating series and the terms \( a_n \) go to zero then the series converges.

8. Absolute convergence.
   
   A series \( \sum_{n=0}^{\infty} a_n \) converges absolutely if the series \( \sum_{n=0}^{\infty} |a_n| \) converges.
   
   If a series \( \sum_{n=0}^{\infty} |a_n| \) converges then so does \( \sum_{n=0}^{\infty} a_n \).

9. The ratio test
   
   If the limit \( \rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \) exists and if less than one, the series converges.
   
   If the limit \( \rho \) exists and is greater than one, the series diverges.

10. The root test
    
    If the limit \( \rho = \lim_{n \to \infty} \sqrt[n]{|a_n|} \) exists and if less than one, the series converges.
    
    If the limit \( \rho \) exists and is greater than one, the series diverges.