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Abstract

This paper characterizes a class of rules for decision-making when an agent knows the possible states of the world and the outcome of each of her actions for each state, but does not have any information about the probabilities of the states. The existing literature in this framework has mainly considered 'max'-based or 'min'-based rules and their variants. Such rules reflect rather extreme forms of optimism or pessimism on the part of an agent. In contrast, this paper focuses on the median outcome(s) and characterizes a class of decision-making rules, which reflects a more 'balanced' attitude towards uncertainty.

Keywords: Non-probabilistic uncertainty, Median outcome, Median-based rules, Complete ignorance.

JEL Classification Number: D81

1. Introduction

This paper characterizes a class of rules for decision-making under the type of non-probabilistic uncertainty considered first by Arrow and Hurwicz (1972). Under this type of uncertainty, the agent knows different possible states of the world and the outcome of each of her actions for each state, but does not have any probabilistic information, such as exact probabilities, the likelihood ranking[1], or probability intervals[2] for these states. Following Arrow and Hurwicz (1972), several writers (see, for example, Maskin (1979), Barrett and Pattanaik (1984), and Barbera and Jackson (1988)), have discussed different rules of decision-making under uncertainty of the Arrow-Hurwicz type. All these contributions, however, focus on 'max'-based or 'min'-based rules and variants of such rules. In light of the agent's usually limited capacity for processing information, it seems intuitively plausible to assume that an agent, when confronted with the problem of choice under uncertainty, may concentrate on some 'focal' outcomes[3] for each action. It is, however, not clear why the agent will necessarily look only at the extreme outcomes, i.e., the best or worst outcomes, of each action. An alternative focal point for each action may be its median outcome(s)[4]. The ranking of actions on the basis of their extreme outcomes involves excessive optimism or pessimism on the part of the agent. In contrast, the focus on the median outcome(s) in ranking alternative actions can be interpreted as a characteristic of more balanced behavior. Though decision rules based on the median outcome(s) seem to have considerable intuitive plausibility, the structure of these rules in the Arrow-Hurwicz framework has not been explored so far. The purpose of this paper is to fill this gap in the literature by providing an axiomatic characterization of a class of median-based decision rules for choice under non-probabilistic uncertainty of the Arrow-Hurwicz type[5].

The structure of the paper is as follows. In section 2, we introduce the basic notation and assumptions. Section 3 presents the axioms with illustrative examples. The main result and its

[1] See Kelsey (1986) for a discussion of decision-making when the agent has only the likelihood ranking of the states, but not their exact probabilities.
[2] See Gilboa and Schmeidler (1989) for a model of decision-making where the agent has a probability interval for each state of the world.
[3] The idea that the agent may consider only some focal outcomes of each available action goes back to Milnor (1954) and Shackle (1954). It may be worth recalling that the paper of Arrow and Hurwicz (1972) was published in a volume in honor of Shackle.
[4] For a precise definition of the median outcome(s) of an action, see Section 2 below.
[5] Nitzan and Pattanaik (1984) characterize a class of median-based decision rules in a framework which was first introduced by Kannai and Peleg (1984), and which is very different from that of Arrow and Hurwicz (1972). Nitzan and Pattanaik, (1984), as well as Kannai and Peleg (1984), do not introduce the states of the world into their model, they assume that the agent knows only the set of outcomes for each action.

proof are given in section 4. Section 5 contains an example of a median-based rule. Finally, section 6 concludes.

2. Notation and Assumptions

Assumption 2.1. The universal set of outcomes, X, is a non-empty and convex subset of $\Re^n$, where n is some fixed positive integer.

Assumption 2.2. The agent has a convex ordering $\succsim$ over X, such that for some $x, y \in X$, $x \succsim y$ and not ($y \succsim x$).

The asymmetric and symmetric factors of $\succsim$ are given by $\succ$ and $\sim$, respectively. Let $d(x, y)$ denote the Euclidean distance between $x, y \in X$.

A decision problem is defined by a (finite) non-empty set of states of the world, *s*. Let Z be the class of all decision problems and let the elements of Z be denoted by S, S', S'' etc. Given $S \in Z$, let $A(S)$ denote the set of all possible functions $a : S \to X$. The elements of $A(S)$ are called actions. For a decision problem $S = \{s_1, ..., s_m\}$, where m is a positive integer, an action specifies exactly one n-tuple of real numbers for each of the m states of the world, and hence, can be thought of as an $m \times n$ vector of real numbers. It may be worth noting here that this representation of actions as $m \times n$ vectors of real numbers allows us later to introduce the property of continuity of the agent's ordering over actions for a given decision problem (see Assumption 2.3 below).

A typical decision problem with two states of the world $S = \{s_1, s_2\}$, two actions $a, b \in A(S)$, and outcomes $a(s_1), a(s_2), b(s_1), b(s_2) \in X$ is described as follows –

$$\begin{array}{ccc} & \multicolumn{2}{c}{S} \\ & s_1 & s_2 \\ a & a(s_1) & a(s_2) \\ b & b(s_1) & b(s_2) \end{array}$$

An action $a \in A(S)$ is *trivial*, if for all $s, s' \in S$, $a(s) = a(s')$. A trivial action $a \in A(S)$, such that for all $s \in S$, $a(s) = x$, is denoted by $[\![x]\!]$.

Assumption 2.3. For all $S \in Z$, the agent has a weak preference ordering R_S defined over $A(S)$, such that,

(i) R_S is continuous[6] over $A(S)$,

and

(ii) for all $x, y \in X$, $x \succsim y$ iff $[\![x]\!] R_S [\![y]\!]$.

I_S and P_S are the symmetric and asymmetric factors, respectively, corresponding to R_S

Remark 2.4. Given Assumption 2.3, the ordering $\succsim$ over *X* is continuous.

For all decision problems $S \in Z$, and, for all $a, b \in A(S)$, we write $a * b$ iff the outcomes of action *a* corresponding to the different states of the world in *S*, constitute a permutation of the outcomes of action *b*.

Let $S = \{s_1, s_2, ..., s_m\} \in Z$ be a decision problem and let $a \in A(S)$. Let the outcomes in the set $A(S)$ be indexed as $x_1, x_2, ..., x_m$ such that for all k $(m > k \geq 1)$, $x_k \succsim x_{k+1}$[7]. Then the set of median outcome(s) of action *a* is denoted by *med(a)* and is defined to be: $\left\{ x_{\frac{m-1}{2}+1} \right\}$ if *m* is odd, and $\left\{ x_{\frac{m}{2}}, x_{\frac{m}{2}+1} \right\}$ if *m* is even.

The agent follows a median-based rule iff for all $S \in Z$ and for all $a, b \in A(S)$, $aI_S b$ if $|med(a)| = |med(b)|$[8] and there exists a one-to-one function *h* from *med(a)* to *med(b)* such that for all $x \in med(a)$, $x \sim h(x)$.

For example, consider the following decision problem $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, actions $a, b \in A(S)$, and outcomes $x_1, x_2, x_2', x_3, x_3', x_3'', y_3, y_4, y_4', y_5, y_6, y_6' \in X$ such that, $x_2' \sim y_4'$, $x_3 \sim y_5$, $x_1 \succsim x_2 \sim x_2' \succsim x_3 \sim x_3' \sim x_3''$, and $y_3 \succsim y_4 \sim y_4' \succsim y_5 \succsim y_6 \sim y_6'$.

S						
	s_1	s_2	s_3	s_4	s_5	s_6
a	x_2'	x_3	x_3'	x_1	x_2	x_3''
b	y_5	y_3	y_4	y_6	y_4'	y_6'

[6] As noted earlier, actions for any given decision problem with *m* states of the world can be thought of as $m \times n$ vectors of real numbers. Therefore, continuity of the agent's ordering over *A*(*S*) can be defined in the usual fashion.
[7] If there are more than one way of indexing the outcomes in this fashion, we choose one of them and keep it fixed.
[8] $|med(a)|$ represents cardinality of the set of median outcome(s) from action *a*.

We choose the indexing of outcomes, such that, $x_1 \succsim x_2 \succsim x_2' \succsim x_3 \succsim x_3' \succsim x_3''$ and $y_3 \succsim y_4 \succsim y_4' \succsim y_5 \succsim y_6 \succsim y_6'$. Then, $med(a) = \{x_2', x_3\}$, $med(b) = \{y_4', y_5\}$, and a median-based rule will yield $aI_S b$.

Note that the class of median-based rules is not necessarily a singleton. Consider the following decision problem $S = \{s_1, s_2\}$ and two actions $a, b \in A(S)$ such that, $b(s_1) \succ a(s_1)$ and $a(s_2) \succ b(s_2)$. It is easy to see that, both $aP_S b$ and $bP_S a$ are consistent with a median-based rule as we have defined it. This shows that a median-based rule need not be unique

	S	
	s_1	s_2
a	$a(s_1)$	$a(s_2)$
b	$b(s_1)$	$b(s_2)$

3. The Axioms

We shall now introduce several plausible properties that the agent may satisfy. The properties are also illustrated with examples. We shall later characterize median-based rules in terms of these properties.

Axiom 3.1. *Neutrality*: Suppose $S, S' \in Z$, $|S| = |S'|$, $a, b \in A(S), a', b' \in A(S')$. Further, suppose there exists a one-to-one function *f* from *S* to S' such that, for all $s, \hat{s} \in S$, $[a(s) \succsim b(\hat{s})$ iff $a'(f(s)) \succsim b'(f(\hat{s}))]$, $[b(\hat{s}) \succsim a(s)$ iff $b'(f(\hat{s})) \succsim a'(f(s))]$, $[a(s) \succsim a(\hat{s})$ iff $a'(f(s)) \succsim a'(f(\hat{s}))]$, $[a(\hat{s}) \succsim a(s)$ iff $a'(f(\hat{s})) \succsim a'(f(s))]$, $[b(s) \succsim b(\hat{s})$ iff $b'(f(s)) \succsim b'(f(\hat{s}))]$, and $[b(\hat{s}) \succsim b(s)$ iff $b'(f(\hat{s})) \succsim b'(f(s))]$. Then [$aR_S b$ iff $a'R_{S'}b'$] and [$bR_S a$ iff $b'R_{S'}a'$].

Suppose two decision problems *S* and S' have equal number of the states of the world. Neutrality then requires that, if the ranking of outcomes from two actions *a* and *b*, in decision problem *S* is "analogous" to the ranking of the outcomes of two actions a' and b', in the decision problem S', then the ranking of *a* and *b* will be similar to the ranking of a' and b'.

For example**,** neutrality implies [$aR_S b$ iff $a'R_{S'}b'$] and [$bR_S a$ iff $b'R_{S'}a'$] in the following two decision problems *S* and S' where $s_1' = f(s_1)$, $s_2' = f(s_2)$, and $p, q, r, x, y, z \in X$ such that $p \succ q \succ r \succ x \succ y \succ z$.

$$
\begin{array}{ccc}
 & S & \\
 & s_1 & s_2 \\
a & r & y \\
b & z & p
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & S' & \\
 & s_1' & s_2' \\
a' & q & x \\
b' & y & p
\end{array}
$$

Neutrality has several plausible implications that have been discussed in the literature for decision-making under complete uncertainty. First, neutrality implies that the identities of the states of a decision problem do not matter while ranking actions in a decision problem; only order of the outcomes under different states matters. Thus, neutrality is similar to the well-known 'symmetry' axiom introduced by Arrow and Hurwicz (1972), but it is stronger than the 'symmetry' axiom. The 'symmetry' axiom as discussed in Arrow and Hurwicz (1972) requires the image set of the mapping from one decision problem to the other to be identical with the domain set, whereas, the image set can be different than the domain set under neutrality.

Second, neutrality implies that, while ranking two actions, only the ranking of outcomes from these two actions are relevant. The ranking of outcomes, at least one of which does not occur in the two actions under consideration, is of no importance. This may be noted as "Independence of the Irrelevant Outcomes".

Thus, in the presence of neutrality, only the ordering of the relevant outcomes under the different states is considered. At first sight, this may seem implausible. Consider the following example with two states and two actions where the outcomes are assumed to be monetary magnitudes.

$$
\begin{array}{ccc}
 & S & \\
 & s_1 & s_2 \\
a & 7 & 1 \\
b & 5 & 5
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & S' & \\
 & s_1' & s_2' \\
a' & 7000 & 4.99 \\
b' & 5 & 5
\end{array}
$$

Suppose, an outcome x is at least as good as y iff $x \geq y$. It is possible for an agent to have $aI_S b$ and $a'P_{S'}b'$, violating neutrality. However, the Arrow-Hurwicz (1972) framework of complete ignorance provides only ordinal information about an agent's preference over the outcomes. Since the ordering of outcomes from $a, b \in A(S)$ is the same across states as the ordering of outcomes from $a', b' \in A(S')$, neutrality seems to be a plausible axiom in this framework.

Lemma 3.2. Suppose the agent satisfies neutrality. Then, for every decision problem $S = \{s_1, s_2, ..., s_m\} \in Z$ and for all actions $\bar{a}, a \in A(S)$, such that, $a * \bar{a}$ and $a(s_1) \succsim a(s_2) \succsim ... \succsim a(s_m)$, we must have $aI_S\bar{a}$.

Proof: Let $S = \{s_1, s_2, ..., s_m\} \in Z$ and let $\bar{a}, a \in A(S)$ such that $a * \bar{a}$ and $a(s_1) \succsim a(s_2) \succsim ... \succsim a(s_m)$. Since $a * \bar{a}$, there exists a one-to-one function *f* from *S* to *S* such that for all $s \in S$, $a(s) = \bar{a}(f(s))$ and hence $a(s) \sim \bar{a}(f(s))$. Therefore, by neutrality [$aR_S\bar{a}$ iff $\bar{a}R_S a$] and [$\bar{a}R_S a$ iff $aR_S\bar{a}$]. Since, by connectedness of R_S, we have ($aR_S\bar{a}$ or $\bar{a}R_S a$), it follows that $aI_S\bar{a}$.

Axiom 3.3. *Duality*: Suppose $S, S' \in Z$, $|S| = |S'|$, $a, b \in A(S)$, $a', b' \in A(S')$, and $s, \hat{s} \in S$. Further, suppose there exists a one-to-one function *g* from *S* to S' such that, for all $s, \hat{s} \in S$, $[a(s) \succsim b(\hat{s})$ iff $b'(g(\hat{s})) \succsim a'(g(s))]$, $[b(\hat{s}) \succsim a(s)$ iff $a'(g(s)) \succsim b'(g(\hat{s}))]$, $[a(s) \succsim a(\hat{s})$ iff $a'(g(\hat{s})) \succsim a'(g(s))]$, $[a(\hat{s}) \succsim a(s)$ iff $a'(g(s)) \succsim a'(g(\hat{s}))]$, $[b(s) \succsim b(\hat{s})$ iff $b'(g(\hat{s})) \succsim b'(g(s))]$, and $[b(\hat{s}) \succsim b(s)$ iff $b'(g(s)) \succsim b'(g(\hat{s}))]$. Then [$aR_S b$ iff $b'R_{S'}a'$] and [$bR_S a$ iff $a'R_{S'}b'$].

Suppose two decision problems *S* and S' have the same number of states of the world. Duality then requires that, if the ranking of outcomes from two actions *a* and *b* in decision problem *S* is the 'reverse' of the ranking of the outcomes of two actions a' and b' in the decision problem S', then the ranking of *a* and *b* must be the 'reverse' of the ranking of a' and b'.

In the following two decision problems *S* and S' such that $s'_1 = g(s_1)$, $s'_2 = g(s_2)$, $s'_3 = g(s_3)$, $p, q, r \in X$ and $p \succ q \succ r$, duality implies [$aR_S b$ iff $b'R_{S'}a'$] and [$bR_S a$ iff $a'R_{S'}b'$]

S

	s_1	s_2	s_2
a	p	q	r
b	r	r	p

S'

	s'_1	s'_2	s'_3
a'	r	q	p
b'	p	p	r

Axiom 3.4. *Weak Dominance*: For all decision problems $S \in Z$, and for all $a, b \in A(S)$, if $a(s) \succ b(s)$ for all $s \in S$, then $aR_S b$.

Thus, if, for every state of the world, an action yields an outcome that is better than the outcome from another action, then the former action is at least as good as the later one. For example, in the following decision problem S, where $a, b \in A(S)$, $p \succsim q \succ r \in X$, weak dominance requires $aR_S b$.

	S	
	s_1	s_2
a	p	q
b	r	r

4. The Main Result

Proposition 4.1. Suppose, Assumptions 2.1 through 2.3 hold. Then the agent follows a median-based rule if she satisfies neutrality, duality, and weak dominance.

We proceed to the proof of Proposition 4.1 via a series of lemmas. Throughout the proof, it is to be understood that Assumptions 2.1, 2.2, and 2.3 hold, and the agent satisfies neutrality, duality, and weak dominance.

Let $S = \{s_1, ..., s_m\} \in Z$ be any decision problem such that $\bar{a}, b, a \in A(S)$, $|med(\bar{a})| = |med(b)|$, there exists a one-to-one function h from $med(\bar{a})$ to $med(b)$ such that for all $x \in med(\bar{a})$, $x \sim h(x)$, $a * \bar{a}$, and $a(s_1) \succsim ... \succsim a(s_m)$. (1)

We assume that (1) holds for the rest of our discussion.

Lemma 4.2: For all $S \in Z$ such that $|S| \leq 2$, and for all $a, b \in A(S)$ such that $b(s) \sim v$ for all $s \in S$, and $\{v\} = med(a)$, we must have $aI_S b$.

Proof: If $|S| = 1$, then, $aI_S b$ follows immediately by reflexivity of R_S. If $|S| = 2$, then, $aI_S b$ follows from reflexivity of R_S and neutrality.

Lemma 4.3: Let $S = \{s_1, ..., s_{2m+1}\} \in Z$ be such that m is a positive integer. Let $a, b \in A(S)$ be such that $a(s_1) \succ a(s_2) \succ ... \succ a(s_{2m+1})$, and $b(s) \sim a(s_{m+1})$ for all $s \in S$. Then, we must have $aI_S b$.

Proof: Consider S and $a,b \in A(S)$ as specified in the statement of lemma 4.3. For the sake of convenience, we represent $a,b,a',b' \in A(S)$ as follows:

$$\begin{array}{cccccc} & & & S & & \\ & s_1 & s_2 & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & s_{2m} & s_{2m+1} \\ a & a(s_1) & a(s_2) & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & a(s_{2m}) & a(s_{2m+1}) \\ b & b(s_1) & b(s_2) & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & b(s_{2m}) & b(s_{2m+1}) \\ a' & a(s_{2m+1}) & a(s_{2m}) & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & a(s_2) & a(s_1) \\ b' & b(s_{2m+1}) & b(s_{2m}) & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & b(s_2) & b(s_1) \end{array}$$

Recall that $b(s) \sim a(s_{m+1})$ for all $s \in S$ and $a(s_1) \succ a(s_2) \succ ... \succ a(s_{2m+1})$. Hence, by neutrality, $aI_S a'$ and $bI_S b'$. By transitivity of R_S, we then have -

($aR_S b$ iff $a'R_S b'$) and ($bR_S a$ iff $b'R_S a'$) (2)

By duality, we have -

($aR_S b$ iff $b'R_S a'$) and ($bR_S a$ iff $a'R_S b'$) (3)

By connectedness of R_S, either $aP_S b$, or $bP_S a$, or $aI_S b$. If $aP_S b$, then, by (2) and (3), $a'P_S b'$ and $b'P_S a'$, which is a contradiction. Similarly $bP_S a$ yields a contradiction. Thus we must have $aI_S b$.

Lemma 4.4: Let $S = \{s_1,...,s_{2m}\} \in Z$ be such that *m* is a positive integer. Let $a,b \in A(S)$ be such that $a(s_1) \succ a(s_2) \succ ... \succ a(s_m) \succsim a(s_{m+1}) \succ a(s_{m+2}) \succ ... \succ a(s_{2m})$, [$b(s_i) \sim a(s_m)$ for $i = 1,...,m$], and [$b(s_i) \sim a(s_{m+1})$ for $i = m+1,...,2m$]. Then we must have $aI_S b$.

Proof: The proof follows exactly similar logic as described in the proof of Lemma 4.3.

Lemma 4.5. Let $x, y \in X$ be such that $x \succ y$ and let $\tilde{\varepsilon}$ be such that $d(x,y) > \tilde{\varepsilon} > 0$. Then, for every positive integers *m*, there exist $w_1, w_2, ..., w_m \in]x,y[$ such that $x \succ w_1 \succ w_2 \succ ... \succ w_m \succ y$, and $\tilde{\varepsilon} > d(w_1,y) > d(w_2,y) > ... > d(w_m,y)$.

Proof: We first show that,

for all $q,r \in X$ such that $q \succ r$, and all $\tilde{\varepsilon}$ such that $d(q,r) > \tilde{\varepsilon} > 0$, there exists $w \in]q,r[$ such that $\tilde{\varepsilon} > d(w,r) > 0$, and $q \succ w \succ r$. (4)

Let $q,r \in X$ be such that, $q \succ r$. Let $\tilde{\varepsilon}$ be such that, $d(q,r) > \tilde{\varepsilon} > 0$.

Since $q \succ r$, by convexity of $\succsim$, $w \succ r$ for all $w \in]q,r[$. (5)

Noting $q \succ r$, by the continuity of $\succsim$, there exists $w \in]q,r[$ such that $\tilde{\varepsilon} > d(w,r) > 0$ and $q \succ w$. (6)

(4) follows from (5) and (6).

Now, let $x,y \in X$ such that $x \succ y$, let $\tilde{\varepsilon}$ be such that $d(x,y) > \tilde{\varepsilon} > 0$, and let *m* be any positive integer.

Since $x,y \in X$, $x \succ y$, and $d(x,y) > \tilde{\varepsilon} > 0$, by (4), there exists $w_1 \in]x,y[$, such that $\tilde{\varepsilon} > d(w_1,y) > 0$, and $x \succ w_1 \succ y$. (7)

Since $w_1, y \in X$, $w_1 \succ y$, and $d(w_1,y) > \varepsilon' > 0$, for some positive ε', by (4) again, there exists $w_2 \in]w_1,y[$ such that, $\varepsilon' > d(w_2,y) > 0$, and $w_1 \succ w_2 \succ y$. (8)

Thus, we have $w_1, w_2 \in]x,y[$ such that, $d(x,y) > \tilde{\varepsilon} > d(w_1,y) > \varepsilon' > d(w_2,y)$, and $x \succ w_1 \succ w_2 \succ y$. Continuing in this fashion, for all $x,y \in X$ such that, $x \succ y$, all $\tilde{\varepsilon}$ such that $d(x,y) > \tilde{\varepsilon} > 0$, and every positive integers *m*, there exist $w_1, w_2, ..., w_m \in]x,y[$ such that $x \succ w_1 \succ w_2 \succ ... \succ w_m \succ y$, and $\tilde{\varepsilon} > d(w_1,y) > d(w_2,y) > ... > d(w_m,y)$.

Lemma 4.3 showed that, in a decision problem with an odd number of states of the world, if two actions *a* and *b* are such that *b* always yields outcomes that are indifferent to the median outcome of *a* and if *a* does not yield indifferent outcomes for any two distinct states of the world, then *a* and *b* must be indifferent. Our next lemma, Lemma 4.6, extends lemma 4.3 by relaxing the requirement that *a* does not yield indifferent outcomes for any two distinct states of the world. Lemma 4.7 extends Lemma 4.4 in an analogous fashion.

Lemma 4.6: Let $S = \{s_1, ..., s_{2m+1}\} \in Z$ be such that *m* $(m \geq 1)$ is a positive integer. Let $a,b \in A(S)$ be such that, $a(s_1) \succsim a(s_2) \succsim ... \succsim a(s_{2m+1})$, and $b(s) \sim a(s_{m+1})$ for all $s \in S$. Then, we must have $aI_S b$.

Proof: Consider $S = \{s_1,...,s_{2m+1}\} \in Z$. Let $a,b \in A(S)$ be such that $a(s_1) \succsim a(s_2) \succsim ... \succsim a(s_{2m+1})$, and $b(s) \sim a(s_{m+1})$ for all $s \in S$. Now, partition S into $S_1, S_2,..., S_t$ such that, [for all $j \in \{1,...,t\}$, and all $s,s' \in S_j$, $a(s) \sim a(s')$], and [for all $j \in \{1,...,t-1\}$, all $s \in S_j$, and all $s' \in S_{j+1}$, $a(s) \succ a(s')$][9]. For all $j \in \{1,...,t\}$, let $m(j)$ be the cardinality of S_j.

By our assumption, there exist $y_0, y \in X$ such that $y_0 \succ y$. Then, by Lemma 4.5, there exists $y_1,..., y_t \in]y_0, y[$ such that, $y_0 \succ y_1 \succ y_2 \succ ... \succ y_t \succ y$ and $d(y_0, y) > d(y_1, y) > d(y_2, y) > ... > d(y_t, y)$. Let ε be a positive number such that $\varepsilon = \min(d(y_0, y_1), d(y_1, y_2),..., d(y_{t-1}, y_t))$. Consider (ε / k), where k ($k \geq 2$) is any positive integer. Then, for every $j \in \{1,...,t\}$, by Lemma 4.5, there exist $w_{j,1}, w_{j,2},..., w_{j,m(j)}$ such that, $y_{j-1} \succ w_{j,1} \succ w_{j,2} \succ ... \succ w_{j,m(j)} \succ y_j$, and $(\varepsilon / k) > d(w_{j,1}, y_j) > d(w_{j,2}, y_j) > ... > d(w_{j,m(j)}, y_j)$.

Let $a_{\varepsilon/k}$ be an action such that for every $j \in \{1,...,t\}$, and $n \in \{1,2,...,m(j)\}$, $a(s_{m(1)+m(2)+...+m(j-1)+n}) = w_{j\,n}$. It is clear by lemma 4.5 that for all $s \in S$, $a_{\varepsilon/k}(s) \succ a(s)$, and $(\varepsilon / k) > d(a_{\varepsilon/k}(s), a(s)) > 0$.

Hence, as $k \to \infty$, $a_{\varepsilon/k}$ converges to action *a*. (9)

Further, note that, for every *k*, $V_k \succ V$, where $\{V_k\} = med(a_{\varepsilon/k})$ and $\{V\} = med(a)$. Then, by Lemma 4.2, $a_{\varepsilon/k} I_S [\![V_k]\!]$ for every *k*[10]. Again, by Assumption 2.3, $[\![V_k]\!] P_S [\![V]\!]$. Hence, by transitivity of R_S, $a_{\varepsilon/k}\, p_S [\![V]\!]$ for all *k*.

Finally, noting (9) and $a_{\varepsilon/k}\, p_S [\![V]\!]$, we have $aR_S [\![V]\!]$, by continuity of R_S. (10)

All that remains to be shown is that [not $aP_S [\![V]\!]$], which given (10), will give us $aI_S [\![V]\!]$. Suppose $aP_S [\![V]\!]$. Then, by continuity of R_S, there exists large enough k', such that, $aP_S [\![V_{k'}]\!]$. But, given $a_{\varepsilon/k} I_S [\![V_k]\!]$ for every *k*, we must have $a_{\varepsilon/k'} I_S [\![V_{k'}]\!]$. Therefore, by transitivity of R_S, it follows that, for some k', $aP_S a_{\varepsilon/k'}$. This contradicts weak dominance, since, as we noted earlier, $a_{\varepsilon/k}(s) \succ a(s)$ for all $s \in S$. This completes the proof of Lemma 4.6.

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[9] If $t = 1$, then $a,b \in A(S)$ are trivial actions such that, Lemma 4.6 follows immediately, by reflexivity of R_S.

[10] Note, that as $k \to \infty$, $[\![V_k]\!]$ converges to $[\![V]\!]$.

Lemma 4.7: Let $S=\{s_1,...,s_{2m}\}\in Z$ be such that m ($m>1$) is a positive integer. Let $a,b\in A(S)$ be such that, $a(s_1)\succsim a(s_2)\succsim...\succsim a(s_{m/2})\succsim a(s_{(m/2)+1})\succsim a(s_{(m/2)+2})\succsim...\succsim a(s_{2m})$, [$b(s_i)\sim a(s_{m/2})$ for $i=1,...,m/2$], and [$b(s_i)\sim a(s_{(m/2)+1})$ for $i=(m/2)+1,...,2m$]. Then, we must have aI_Sb.

Proof: Consider $S=\{s_1,...,s_{2m}\}\in Z$ such that m ($m>1$) is a positive integer. Let $a,b\in A(S)$ be such that, $a(s_1)\succsim a(s_2)\succsim...\succsim a(s_{m/2})\succsim a(s_{(m/2)+1})\succsim a(s_{(m/2)+2})\succsim...\succsim a(s_{2m})$, [$b(s_i)\sim a(s_{m/2})$ for $i=1,...,m/2$], and [$b(s_i)\sim a(s_{(m/2)+1})$ for $i=(m/2)+1,...,2m$]. Now, partition S into $S_1,S_2,...,S_t$ such that, [for all $j\in\{1,...,t\}$, and all $s,s'\in S_j$, $a(s)\sim a(s')$], and [for all $j\in\{1,...,t-1\}$, all $s\in S_j$, and all $s'\in S_{j+1}$, $a(s)\succ a(s')$]. The rest of the proof is similar to the proof of Lemma 4.6.

Proof of Proposition 4.1: Finally, given (1) and Lemma 3.2, we have $aI_S\bar{a}$. Thus, by transitivity of R_S, and following Lemmas 4.2, 4.3, 4.6, and 4.7, we must have $\bar{a}I_Sb$.

5. An Example

We have shown that, given Assumptions 2.1, 2.2, and 2.3, if the agent satisfies neutrality, duality, and weak dominance, she must follow a median-based rule. We now give an example where Assumptions 2.1, 2.2, and 2.3 as well as neutrality, duality, and weak dominance are all satisfied.

Let X be any non-empty and convex subset of $\Re^n$ and let $\succsim$ be any convex and continuous ordering over X, such that, for some $x,y\in X$, $x\succ y$. Thus, by our specification, Assumptions 2.1, 2.2, and 2.3 (i) are satisfied. Let U be a real valued and continuous utility function representing $\succsim$. Clearly, such a utility function U exists[11]. For every decision problem $S\in Z$, let R_S defined over $A(S)$ be such that, for all $a,b\in A(S)$, aR_Sb iff $\sum_{x\in med(a)}U(x)\geq\sum_{y\in med(b)}U(y)$. Clearly, for every $S\in Z$, R_S is continuous and for all $x,y\in X$, $[\![x]\!]R_S[\![y]\!]$ iff $x\succsim y$, and hence Assumption 2.3 (ii) is satisfied. Further, for all $S\in Z$, and for all

[11] See Debreu (1959).

$a,b \in A(S)$, if $|med(a)| = |med(b)|$ and there exists a one-to-one function h from *med(a)* to *med(b)* such that for all $x \in med(a),\ x \sim h(x)$, then $\sum_{x \in med(a)} U(x) = \sum_{y \in med(b)} U(y)$ and hence $aI_S b$.

6. Concluding Remarks

Most of the papers, which discuss non-probabilistic uncertainty of the Arrow-Hurwicz type, focus on what may be called positional decision rules. The positional rules characterized in this literature mainly consider the best or the worst outcomes. Lexicographic variants of such rules have also been discussed. It is, however, surprising that none of the papers in this area have dealt with the case when the agent makes decision on the basis of the median outcome(s) of her actions. In this paper, we have sought to fill this gap by providing an axiomatic characterization of median-based rules.

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