Estimates of norms of subsequent derivatives of r-monotone functions.

joint work with V. Babenko

Statement of the Kolmogorov problem.

Let $G = \mathbb{R}$ or $G = \mathbb{R}_-$.

Let some class $X \subset L^r_{\infty,\infty}(G)$ and an arbitrary system of integers

$$0 \le k_0 < k_1 < ... < k_d = r$$

be given. The problem is to find the necessary and sufficient conditions for the system of positive numbers

$$M_{k_0}, M_{k_1}, ..., M_{k_d}$$

to guarantee the existence of the function $x \in X$ such that

$$\|x^{(k_i)}\| = M_{k_i}, \qquad i = 0, ..., d.$$

History.

Kolmogorov (1939): for any <u>three</u> positive $M_0, M_k, M_r,\ 0 < k < r$, $\exists x \in L^r_{\infty,\infty}(\mathbb{R})$ such that

$$\|x\| = M_0, \qquad \|x^{(k)}\| = M_k, \qquad \|x^{(r)}\| = M_r,$$

if and only if

$$M_k \le \frac{\|\varphi_{r-k}\|}{\|\varphi_r\|^{1-k/r}} M_0^{1-k/r} M_r^{k/r},$$

where φ_r is the r^{th} periodic integral with zero mean value on a period of the function $\varphi_0(t) = \operatorname{sgn}\sin t$.

Case $\mathbb{R}$

The only other known solutions for the case $G = \mathbb{R}$ are:

1. $X = L^r_{\infty,\infty}(\mathbb{R})$; $k_0 = 0,\ k_1 = r-2,\ k_2 = r-1,\ k_3 = r$ (Rodov 1946)
2. $X = L^r_{\infty,\infty}(\mathbb{R})$; $k_0 = 0 < k_1 < k_2 = r-2,\ k_3 = r-1,\ k_4 = r$ (Rodov 1954)
3. $X = L^r_{\infty,\infty}(\mathbb{R})$; $k_0 = 0 < k_1 < k_2 = r-1,\ k_3 = r$ (Dzyadyk, Dubovik 1975)

Rodov found sufficient conditions for cases

$$M_0, M_1, M_2, M_5 \ \text{ and } \ M_0, M_1, M_2, M_3, M_4, M_5.$$

As for the general case of arbitrary d the only result is by Dzyadyk and Dubovik. The result provides some sufficient conditions for the function $x \in L^r_{\infty,\infty}(\mathbb{R})$ to exist.

Remark. Note that only one intermediate derivative (of order k_1) is arbitrary, the other one (of order k_2) is related to r.

Case $\mathbb{R}_-$

In the case $G = \mathbb{R}_-$, solutions of the Kolmogorov problem are:

1. $X = L^r_{\infty,\infty}(\mathbb{R}_-)$, $k_0 = 0 < k_1 < k_2 = r$ (follows from the results of Landau 1913, Matorin 1955, and Schoenberg and Cavaretta 1970)

2. $X = L^r_{\infty,\infty}(\mathbb{R}_-)$, $k_0 = 0 < k_1 < k_2 = r-1$, $k_3 = r$ (V. Babenko and Britvin 2002)

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3. $X = L^{r,r}_{\infty,\infty}(\mathbb{R}_-)$, $k_0 = 0 < k_1 < k_2 = r$ (Olovyanishnikov 1951)

4. $X = L^{r,r}_{\infty,\infty}(\mathbb{R}_-)$, $k_0 = 0 < k_1 < k_2 = r-1$, $k_3 = r$ (Yattselev 1999)

Remark. As before, only one intermediate derivative (of order k_1) is arbitrary, the other one (of order k_2) is related to r.

Kolmogorov problem for four numbers.

For any $a > b > 0$ and $l > 0$, set

$$\phi_r(a, l; t) := \frac{l}{r!}(t + a)_+^r,$$

where $u_+ = \max\{u, 0\}$, and

$$\phi_r(a, b, l; t) := \phi_r(a, l; t) - \phi_r(b, l; t).$$

It is not hard to show that there exist values of parameters $a = \tilde{a}$, $b = \tilde{b}$, and $l = \tilde{l}$ such that

$$\|\phi_r^{(k)}(\tilde{a}, \tilde{b}, \tilde{l}; \cdot)\| = M_k, \qquad k = k_1, k_2, r.$$

Set

$$\Phi_r(M_{k_1}, M_{k_2}, M_r; t) := \phi_r(\tilde{a}, \tilde{b}, \tilde{l}; t).$$

Theorem 1. *(Necessary conditions)*

Let $k_1, k_2, r \in \mathbb{N}$, $0 < k_1 < k_2 < r$, *and positive numbers* $M_0, M_{k_1}, M_{k_2}, M_r$ *be given.*

If there is a function $x \in L^{r,r}_{\infty,\infty}(\mathbb{R}_-)$ *such that*

$$\|x^{(k)}\| = M_k, \qquad k = 0, k_1, k_2, r,$$

then the following inequalities are valid:

$$(1) \qquad \begin{cases} a) \quad M_{k_2} \;\le\; \dfrac{(r-k_1)!^{\frac{r-k_2}{r-k_1}}}{(r-k_2)!} M_{k_1}^{\frac{r-k_2}{r-k_1}} M_r^{\frac{k_2-k_1}{r-k_1}}, \\[2ex] b) \quad M_{k_2} \;\le\; \dfrac{(r)!^{\frac{r-k_2}{r}}}{(r-k_2)!} M_0^{\frac{r-k_2}{r}} M_r^{\frac{k_2}{r}}, \\[2ex] c) \quad M_{k_1} \;\le\; \dfrac{(r)!^{\frac{r-k_1}{r}}}{(r-k_1)!} M_0^{\frac{r-k_1}{r}} M_r^{\frac{k_1}{r}}; \end{cases}$$

$$(2) \qquad \begin{cases} a) \quad M_0 \;\ge\; \Phi_r\left(M_{k_1}, M_{k_2}, M_r; 0\right), \\ b) \quad M_{k_1} \;\le\; \Phi_r^{(k_1)}(M_0, M_{k_2}, M_r; 0), \\ c) \quad M_{k_2} \;\ge\; \Phi_r^{(k_2)}(M_0, M_{k_1}, M_r; 0). \end{cases}$$

Main ingredients of the proof.

- We use comparison theorem to prove the analog of Kolmogorov-Olovyanishnikov inequality for four numbers.

- We answer the following question: if we fix the value of the last and any other two derivatives, what can be said about the remaining one?

Theorem 2. *(Sufficient conditions)*

Let $k_1, k_2, r \in \mathbb{N}$, $0 < k_1 < k_2 < r$, *and positive numbers* $M_0, M_{k_1}, M_{k_2}, M_r$ *be given.*

If inequalities (1) are valid together with any of inequalities (2), then there is a function $x \in L^{r,r}_{\infty,\infty}(\mathbb{R}_-)$ *such that*

$$\|x^{(k)}\| = M_k, \qquad k = 0, k_1, k_2, r.$$

Kolmogorov Problem for five numbers

The following theorem gives necessary conditions on five positive numbers

$$M_0, \quad M_{k_1}, \quad M_{k_2}, \quad M_{k_3}, \quad M_r$$

to solve the Kolmogorov Problem.

Theorem 3. *1.* M_{k_2}, M_{k_3}, M_r *satisfy the Olovyanishnikov inequality:*

$$M_{k_3} \le \frac{(r-k_2)^{\frac{r-k_3}{r-k_2}}}{(r-k_3)!} M_{k_2}^{\frac{r-k_3}{r-k_2}} M_r^{\frac{k_3-k_2}{r-k_2}} \tag{1}$$

2.(a) If inequality (1) is satisfied with "=", then there exists a unique spline of type I (call it ϕ_1*) such that*

$$\begin{aligned} \|\phi_1^{(k_2)}\| &= M_{k_2} \\ \|\phi_1^{(k_3)}\| &= M_{k_3} \\ \|\phi_1^{(r)}\| &= M_r \end{aligned} \tag{2}$$

then the necessary conditions are

$$\begin{aligned} M_{k_1} &= \|\phi_1^{(k_1)}\| \\ M_0 &\ge \|\phi_1\| \end{aligned} \tag{3}$$

(b) *If inequality (1) is satisfied with "$<$", then there exists a unique spline of type II (call it ϕ_2) such that*

$$\begin{aligned} \|\phi_2^{(k_2)}\| &= M_{k_2} \\ \|\phi_2^{(k_3)}\| &= M_{k_3} \\ \|\phi_2^{(r)}\| &= M_r \end{aligned} \tag{4}$$

then the solution of KP for four numbers implies that

$$M_{k_1} \geq \|\phi_2^{(k_1)}\|. \tag{5}$$

If (5) occurs with "$=$" then the necessary condition is

$$M_0 \geq \|\phi_2\|. \tag{6}$$

If (5) occurs with "$<$" then there exists a unique spline of type III (call it ϕ_3) such that

$$\begin{aligned} \|\phi_3^{(k_1)}\| &= M_{k_1} \\ \|\phi_3^{(k_2)}\| &= M_{k_2} \\ \|\phi_3^{(k_3)}\| &= M_{k_3} \\ \|\phi_3^{(r)}\| &= M_r \end{aligned} \tag{7}$$

and then the necessary condition is

$$M_0 \geq \|\phi_3\|. \tag{8}$$

Obviously, these necessary conditions are sufficient as well.

For an arbitrary d...

For $l > 0$ and $0 < a_1 < a_2 < \cdots < a_{d-2}$ define

$$\phi_r(t) = \phi_r(a_1, \ldots, a_{d-2}, l; t) := \phi_r\left(\sum_{j=1}^{d-2} a_j, l; t\right) + \sum_{i=1}^{d-3} (-1)^i \phi_r\left(\sum_{j=1}^{d-2-i} a_j, l; t\right).$$

Theorem 4. *Let d and $0 < k_1 < k_2 < \cdots < k_{d-1} < k_d = r$ be given integers. If $x \in L^{r,r}_{\infty,\infty}(\mathbb{R}_-)$ and parameters of ϕ_r are chosen so that*

$$\forall i = 0, 1, \ldots, d-2, \qquad \textit{and} \qquad i \neq j \in \{0, 1, \ldots, d-2\}$$

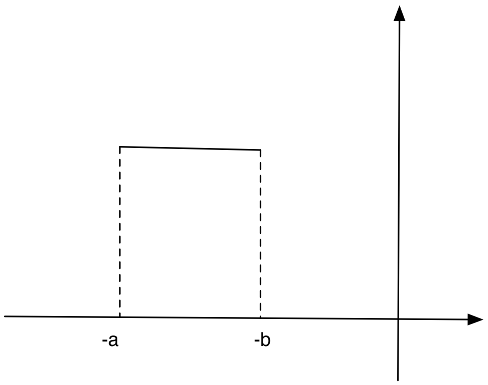
$$(-1)^i \|\phi_r^{(k_i)}\| \geq (-1)^i \|x^{(k_i)}\|,$$

$$\textit{and} \quad \|\phi_r^{(r)}\| \geq \|x^{(r)}\|,$$

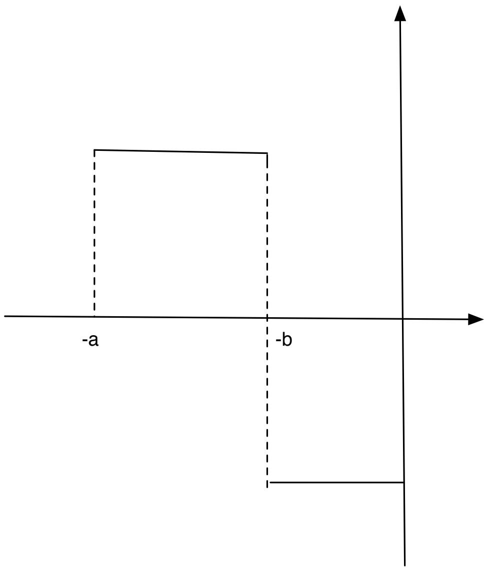
then $\forall k_{j-1} < k < k_{j+1}$

$$(-1)^k \|\phi_r^{(k)}\| \leq (-1)^k \|x^{(k)}\|.$$

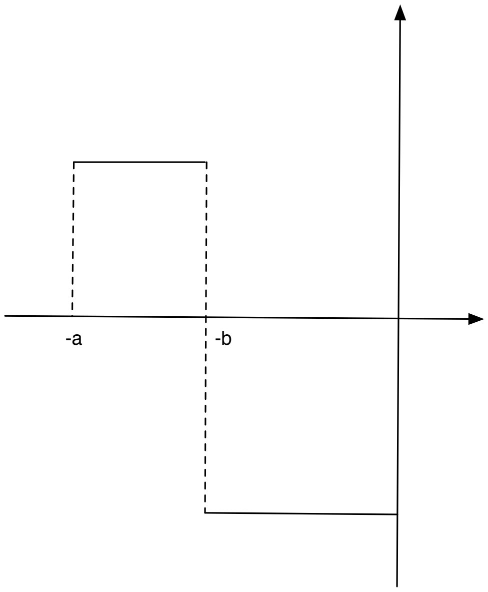
Discussion of the extremal function for four numbers



For the class $X = L^{r,r}_{\infty,\infty}(\mathbb{R}_-)$*:*



For the class $X = L^{r,r}_{\infty,\infty}(\mathbb{R}_-)$*:*



For the class $X = L^{r,r}_{\infty,\infty}(\mathbb{R}_-)$*:*

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